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UNIVERSITY OF INNSBRUCK

Ultracold Atoms and the Functional Renormalization Group

IQOQI
AUSTRIAN ACADEMY OF SCIENCES

SFB
Coherent Control of Quantum Systems

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Introduction: Many-Body Physics with Cold Atoms

Q: What can cold atoms add to many body physics?

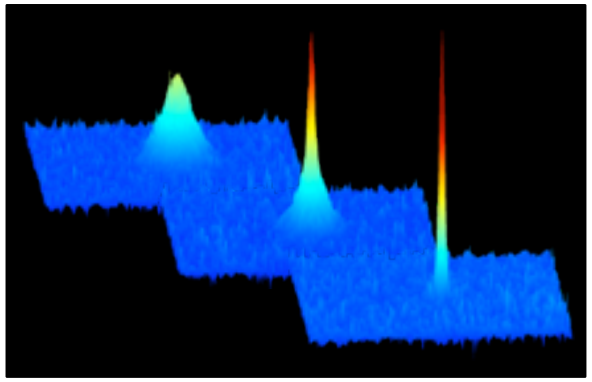
- New models of own interest
  - Bose-Hubbard model
  - Strongly interacting continuum systems: BCS-BEC Crossover; Efimov effect
  - systems with long range interactions (polar molecules, Rydberg atoms), eg. 1/r^3
- Quantum Simulation: clean/ controllable realization of model Hamiltonians which are
  - less clear to what extent realized in condensed matter
  - extremely hard to analyze theoretically

  e.g. 2d Fermi-Hubbard model
- Nonequilibrium Physics of closed systems: time dependence
  - Condensed matter: fast equilibration, thermodynamic equilibrium stationary state physics.
  - Cold atoms: study dynamical evolution, e.g, quench dynamics, thermalization dynamics
- Nonequilibrium Physics of open systems: Driven-dissipative many-body equilibria
  - go beyond coherent manipulation of many-body systems: add drive and controlled dissipation
  - merge techniques from quantum optics and many-body physics

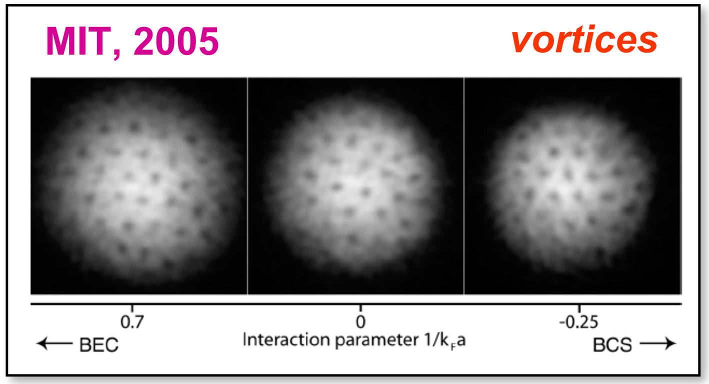
Lecture Outline

- Here we concentrate on one of these key aspects: The transition to macrophysics starting from well-controlled, clean microphysics

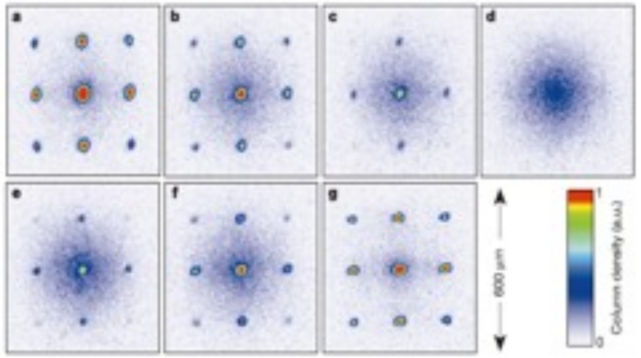
- Continuum systems:
  - Scales and interactions, Effective theories for atomic gases
  - The cornerstones of quantum condensation phenomena:
    - Weakly interacting Bosons, Bose-Einstein condensation
    - Weakly interacting Fermions, BCS instability
  - Synthesis: Strong interactions, the BCS-BEC crossover in Functional RG framework
    - Basic picture: The crossover phase diagram
    - Closer look at various scales: from scattering amplitudes to critical behavior
- Lattice systems:
  - The Bose-Hubbard model in optical lattices
  - Phase Diagram: Mott insulator - superfluid transition
  - FRG approach to strongly correlated lattice systems
  - BCS-BEC analog for bosons on the lattice: Ising type quantum phase transition



Bose-Einstein Condensation



Fermion Superfluidity



Mott insulator - superfluid transition

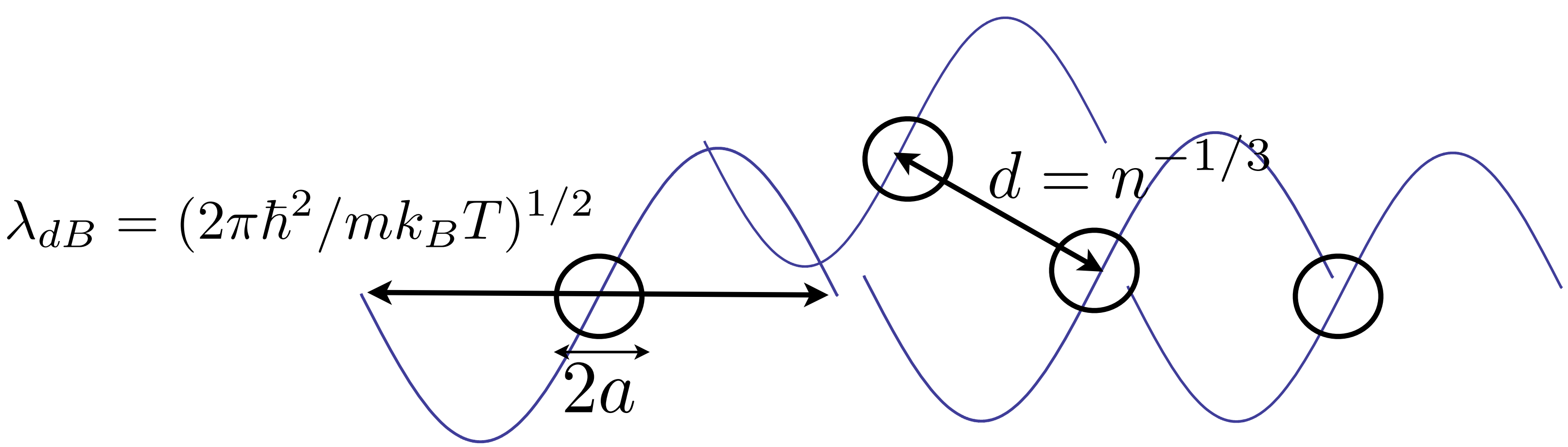
Literature

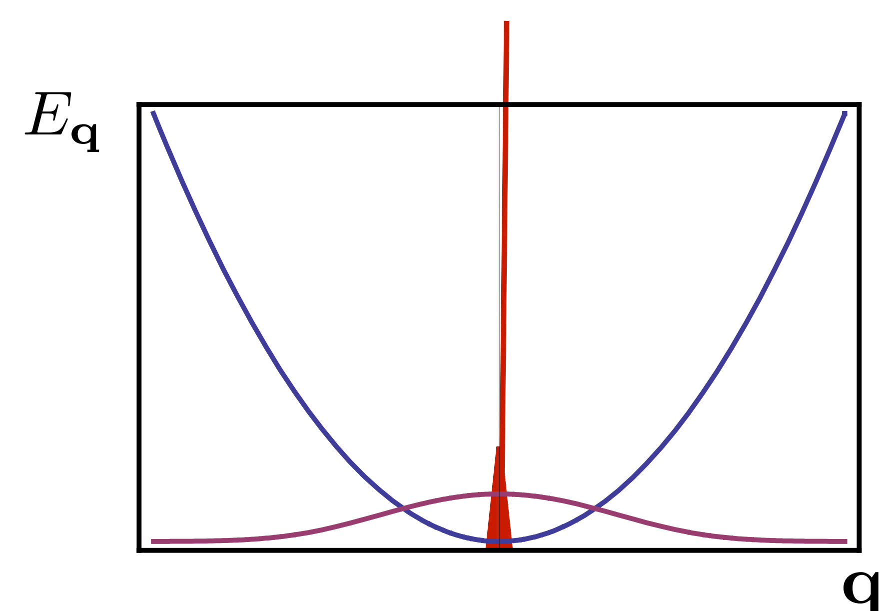
- General BEC/BCS theory (introductory):
  - C. J. Pethick, H. Smith, Bose-Einstein Condensation in Dilute Gases, Cambridge University Press (2002)
  - L. Pitaevski and S. Stringari, Bose-Einstein Condensation, Oxford University Press (2003)

- Recent research review articles:
  - I. Bloch, J. Dalibard and W. Zwerger, Many-Body Physics with Cold Atoms, Rev. Mod. Phys. **80**, 885 (2008)
  - S. Giorgini, L. Pitaevski and S. Stringari, Theory of ultracold atomic Fermi gases, Rev. Mod. Phys. **80**, 1215 (2008)

- Many-Body Physics (with and without cold atoms)
  - J. Negele, H. Orland, Quantum Many-particle Systems, Westview Press (1998)
  - A. Altland and B. Simons, Condensed Matter Field Theory, Cambridge University Press (2006, new edition 2010)
  - S. Sachdev, Quantum Phase Transitions, Cambridge University Press (1999)

Scales and Interactions in Ultracold Quantum Gases





Hamiltonian for weakly interacting ultracold bosonic atoms

- Our workhorse Hamiltonian is

$$H = H_{kin} + H_{trap} + H_{int}$$

- With ingredients:
  - Kinetic energy: motion of nonrelativistic particles

$$H_{kin} = \int_{\mathbf{x}} a^\dagger_{\mathbf{x}} \big(-\frac{\triangle}{2m} \big) a_{\mathbf{x}} = \int_{\mathbf{q}} a^\dagger_{\mathbf{q}} \big(\frac{\mathbf{q}^2}{2m} \big) a_{\mathbf{q}}$$

after Fourier transform $\quad a_{\mathbf{q}} = \int_{\mathbf{x}} e^{i\mathbf{q}\mathbf{x}} a_{\mathbf{x}}; \quad [a_{\mathbf{x}}, a^\dagger_{\mathbf{y}}] = \delta(\mathbf{x} - \mathbf{y})$ (bosons)

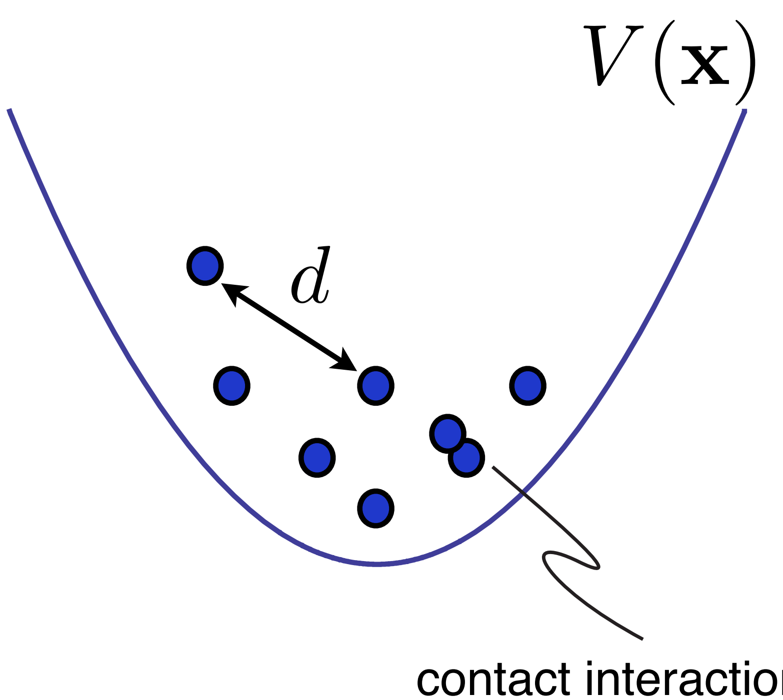
  - Trapping potential: the local density experiences a local potential energy

$$H_{trap} = \int_{\mathbf{x}} V(\mathbf{x}) \hat{n}_{\mathbf{x}}, \quad V(\mathbf{x}) = \tfrac{1}{2} m \omega^2 \mathbf{x}^2$$

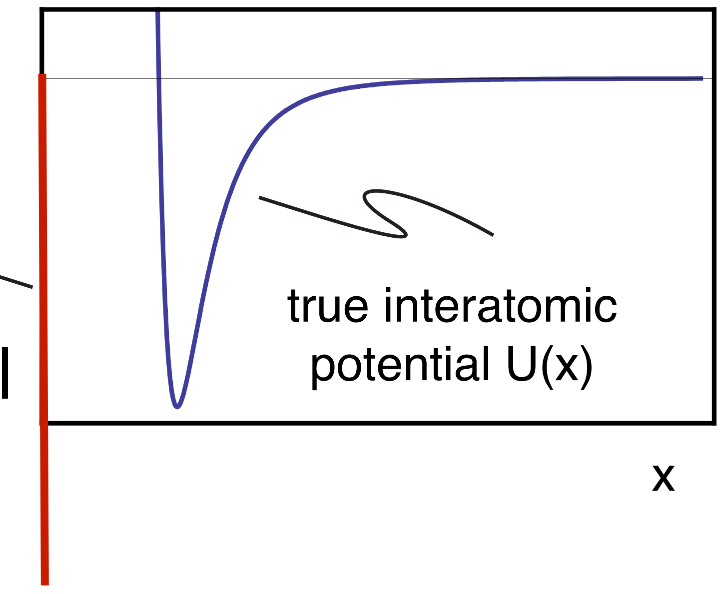
$\hat{n}_{\mathbf{x}} = \hat{a}^\dagger_{\mathbf{x}} \hat{a}_{\mathbf{x}}$ density operator

  - Local two-body interactions:

$$H_{int} = \int_{\mathbf{x},\mathbf{y}} g \delta(\mathbf{x} - \mathbf{y}) \hat{n}_{\mathbf{x}} \hat{n}_{\mathbf{y}} = g \int_{\mathbf{x}} \hat{n}^2_{\mathbf{x}}$$



Microscopic Origin of the Interaction Term



- Microscopic scattering physics: Lennard-Jones (LJ) potential
  - $1/r^{12}$ hard core repulsion: repulsion of electron clouds $r_{rep} = \mathcal{O}(a_B)$
  - $1/r^6$ attraction: van der Waals (induced dipole-dipole interaction)
    $r_{vdW} = (50...200)a_B$ for alkalis: typ. order of magnitude for interaction length scale

- General properties of LJ type potentials at low energies:
  - isotropic s-wave scattering dominates; the scattered wave function behaves asymptotically as $\psi(\mathbf{x}) \sim a/\mathbf{x}$
  - a is the scattering length. Knowledge of this single parameter is sufficient to describe low energy scattering!
  - within Born approximation, it can be calculated as

$$a_{\text{Born}} \sim \int_{\mathbf{x}} U(\mathbf{x})$$

interatomic potential

➡very different interaction potentials may have the same scattering length!

The Model Hamiltonian as an Effective Theory

$$H = \int_{\mathbf{x}} [a^\dagger_{\mathbf{x}}\big(-\frac{\triangle}{2m} + V(x)\big)a_{\mathbf{x}} + g\hat{n}^2_{\mathbf{x}}]$$

- Efficient description by an effective Hamiltonian with few parameters.
- For ultracold bosonic alkali gases, a single parameter, the scattering length a, is sufficient to characterize low energy scattering physics of indistinguishable particles : Effective interaction

$$g = \frac{8\pi\hbar^2}{m}a$$

- A typical order of magnitude for the scattering length is

$$a = \mathcal{O}(r_{vdW}), \;\; r_{vdW}(50...200)a_B$$

- For bosons, we must restrict to repulsive interactions a > 0 (else: bosons seek solid ground state, collapse in real space)
- The validity of the model Hamiltonian is restricted to length scales

$$l \gg r_{vdW}$$

- So far: microscopic description; now: many body scales!

➡ Finite temperature T; finite density n

BEC: Statistical Mechanics of Noninteracting Bosons

- An ensemble of noninteracting bosons in free space is described in the grand canonical ensemble:

total particle number

in general: $$H \to H - \mu \hat{N}$$

free particles: $$H_{kin} \to H_{kin} - \mu \hat{N} = \int_{\mathbf{x}} a^\dagger_{\mathbf{x}} \Big(- \frac{\triangle}{2m} - \mu \Big) a_{\mathbf{x}} = \int_{\mathbf{q}} a^\dagger_{\mathbf{q}} \Big(\frac{\mathbf{q}^2}{2m} - \mu \Big) a_{\mathbf{q}}$$

- Statistical properties described by the Free Energy:

$$U = k_B T \log Z, \quad Z = \mathrm{tr} \exp - \tfrac{1}{k_B T} (H - \mu \hat{N}) \qquad \mu \leq 0$$

chemical potential

temperature

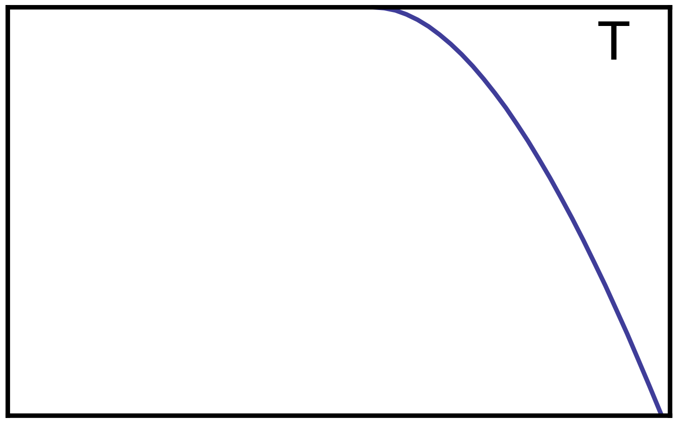
- The chemical potential adjusts the average particle number via the equation of state:

$$N = - \frac{\partial U}{\partial \mu} = \langle \int_{\mathbf{q}} a^\dagger_{\mathbf{q}} a_{\mathbf{q}} \rangle = \int_{\mathbf{q}} \langle a^\dagger_{\mathbf{q}} a_{\mathbf{q}} \rangle = V \int \frac{d^d q}{(2\pi)^d} \frac{1}{e^{\frac{1}{k_B T}(\frac{\mathbf{q}^2}{2m} - \mu)} - 1}$$

Bose-Einstein distribution

Bose-Einstein Condensation (3D)

- Lower T and study the behavior of μ at fixed n (3D):



$$n = \frac{N}{V} = \int \frac{d^d q}{(2\pi)^d} \frac{1}{e^{\frac{1}{k_B T}\left(\frac{\mathbf{q}^2}{2m} - \mu\right)} - 1}$$

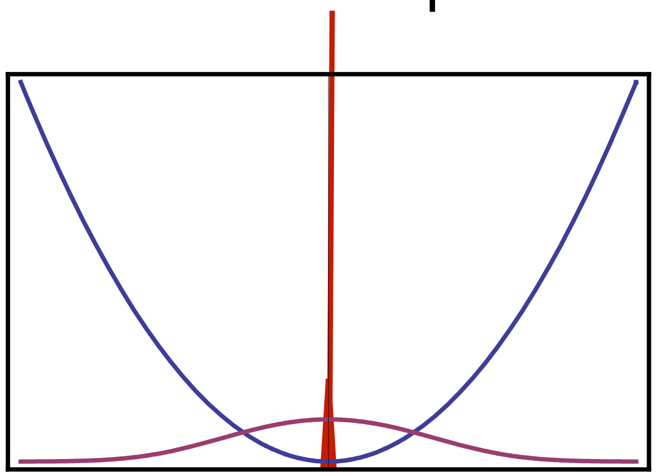
$d = 3$

$$= \lambda_{dB}^{-3}(T) g_{3/2}(e^{\mu/k_B T})$$

Polygamma function

$$g_\alpha(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\alpha}$$

- At a finite T, μ hits zero: below this T_c the equation of state has no solution
- Bose and Einstein (1925): Equation below T_c needs modification due to macroscopic occupation of zero mode:



$$n = \langle a_{\mathbf{0}}^\dagger a_{\mathbf{0}} \rangle + \int \frac{d^d q}{(2\pi)^d} \frac{1}{e^{\frac{\mathbf{q}^2}{2m k_B T}} - 1}$$

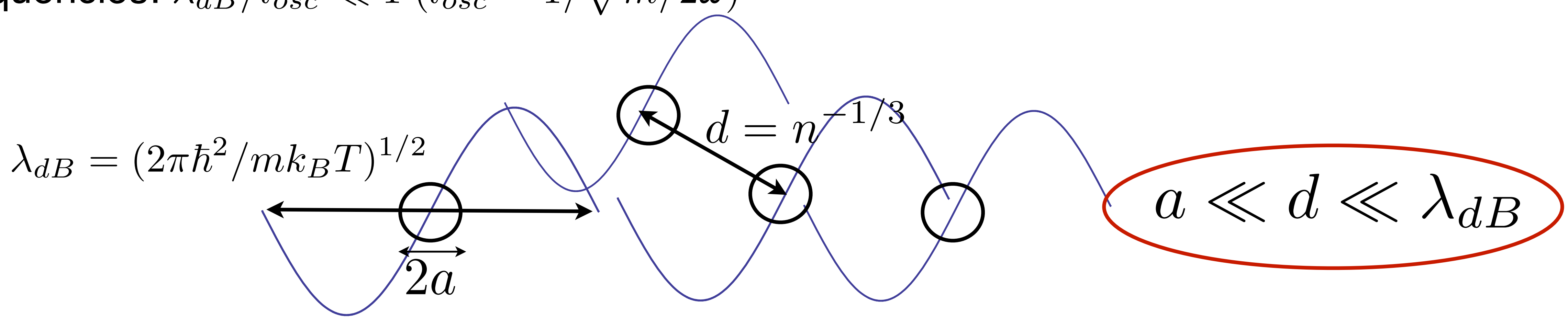
- plausible: Bosons can populate single quantum state with arbitrary number
- macroscopic: $N_0 = \langle a_{\mathbf{0}}^\dagger a_{\mathbf{0}} \rangle = \mathcal{O}(N) \propto V$, i.e. extensive
- critical temperature: determined by

$$n\lambda_{dB}^3 = g_{3/2}(1) = \zeta(3/2) \approx 2.612$$

de Broglie wavelength $\lambda_{dB} = (2\pi\hbar^2/mk_B T)^{1/2} \gtrsim d = n^{-1/3}$ interparticle spacing

Validity of our Hamiltonian: Scales in Cold Dilute Bose Gases

- The effective Hamiltonian is valid because none of many-body length scales can resolve interaction length scale :
- Many-body scales: density and temperature in terms of length scales.
  - diluteness $a/d \ll 1$ $(d = n^{-1/3})$
    - $*$ dilute means weakly interacting: interaction energy $gn \sim a/d \cdot d^{-2}$
    - $*$ clear: three-body interaction terms irrelevant
  - quantum degeneracy: $d/\lambda_{dB} \ll 1$ $(\lambda_{dB} = (2\pi\hbar^2/mk_BT)^{1/2})$
  - trap frequencies: $\lambda_{dB}/l_{osc} \ll 1$ $(l_{osc} = 1/\sqrt{m/2\omega})$



Summary of length scales $\quad a_{\mathrm{B}} = 5.3 \times 10^{-2}\mathrm{nm}$ Bohr radius

length	scattering length a/a_B	interparticle sep. d/a_B	de Broglie w.l. λ_{dB}/a_B	trap size l_{osc}/a_B
	(0.05 ... 0.2)10^3	(0.8 ... 3)10^3	(10 ... 40)10^3	(3 ... 300)10^3

phys. meaning of the ratio:

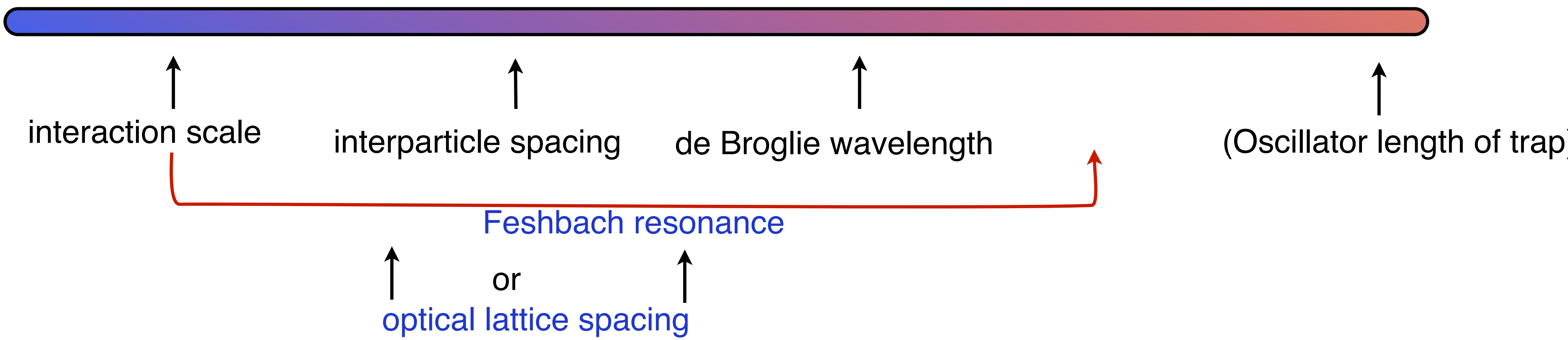
weak interactions/ dilute gases

quantum degeneracy

local density approximation

Violations of the scale hierarchy

- Generic sequence of scales and possible violations:

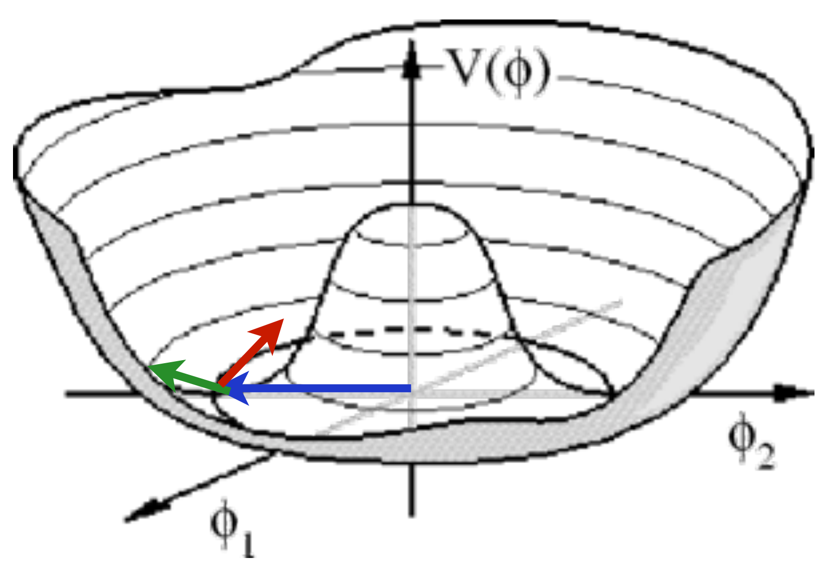


- With Feshbach resonances, violation of a/d << 1 possible: Dense degenerate system
- With optical lattices, a new length and a new energy scale are introduced:
  - lattice spacing = wavelength of light: high densities (“fillings) become available
  - lattice depth: Kinetic energy is withdrawn more strongly than interaction energy: “strong correlations”

- Both leads to the possibility of “strong interactions/correlations” as we will see

- NB: Despite violation of scale hierarchy for dilute quantum gases, we will be able to give accurate microscopic models

Weakly Interacting Bosons



Effective Action

- grand canonical workhorse Hamiltonian (no trap)

$$H[\hat{\phi}(\mathbf{x})^\dagger, \hat{\phi}(\mathbf{x})] = \int_{\mathbf{x}} [\hat{\phi}(\mathbf{x})^\dagger (- \frac{\triangle}{2m} - \mu) \hat{\phi}(\mathbf{x}) + g(\hat{\phi}(\mathbf{x})^\dagger \hat{\phi}(\mathbf{x}))^2]$$

- the associated euclidean classical action for the nonrelativistic problem is $(x = (\tau, \mathbf{x}))$

$$S[\varphi^*(x), \varphi(x)] = \int_0^\beta d\tau \left[\left(\int d^3x \varphi^*(x) \partial_\tau \varphi(x) \right) + H[\varphi^*(x), \varphi(x)] \right]$$

- and the many-body quantum problem can be formulated e.g. in terms of the effective action

$$\boxed{\exp -\Gamma[\phi^*, \phi] = \int \mathcal{D}(\delta\varphi^*, \delta\varphi) \exp -S[\phi^* + \delta\varphi^*, \phi + \delta\varphi], \quad \frac{\delta\Gamma[\phi^*, \phi]}{\delta\phi(x)} = 0}$$

field equation

- Discussion:
  - $\phi = \langle\varphi\rangle$ is the classical field/field expectation value
  - The effective action can be understood “classical action plus fluctuations”. It lends itself to semiclassical approximations (small fluctuations around a mean field)
  - NB: Action principle is leveraged over to full quantum status
  - Symmetry principles are leveraged over from the classical action to full quantum status

Generalities of the Microscopic Action

- The classical action is

$$S[\phi^*,\phi] = \int dt \int d^3x \left[\phi(x)^*(\partial_\tau - \tfrac{\triangle}{2M} - \mu)\phi(x) + \tfrac{g}{2}(\phi^*(x)\phi(x))^2\right]$$

- Symmetries:
  - obviously, Lorentz invariance replaced by Galilei invariance. Different power counting, since $\omega \sim \mathbf{q}^2$: The dynamic exponent $z = 2$ and the canonical dimension of the Lagrangian $d + z = 5$.
  - Unlike relativistic models, the temporal derivative term is a pure phase

$$\Big(\int_\tau \phi(x)^*\partial_\tau\phi(x)\Big)^* = -\int \phi(x)^*\partial_\tau\phi(x)$$

  - i.e. relation to classical statistical model less clear
  - Global phase rotation invariance $U(1)$ with linear time derivative gives particle number conservation
  - A further interesting symmetry is a temporally local gauge invariance

$$\phi(x) \to e^{\mathrm{i}\theta(\tau)}\phi(x),\ \phi^*(x) \to e^{-\mathrm{i}\theta(\tau)}\phi^*(x), \quad \mu \to \mu + \mathrm{i}\theta(\tau)$$

  - with physical consequences: see Bose-Hubbard model!

The Gross-Pitaevski Equation

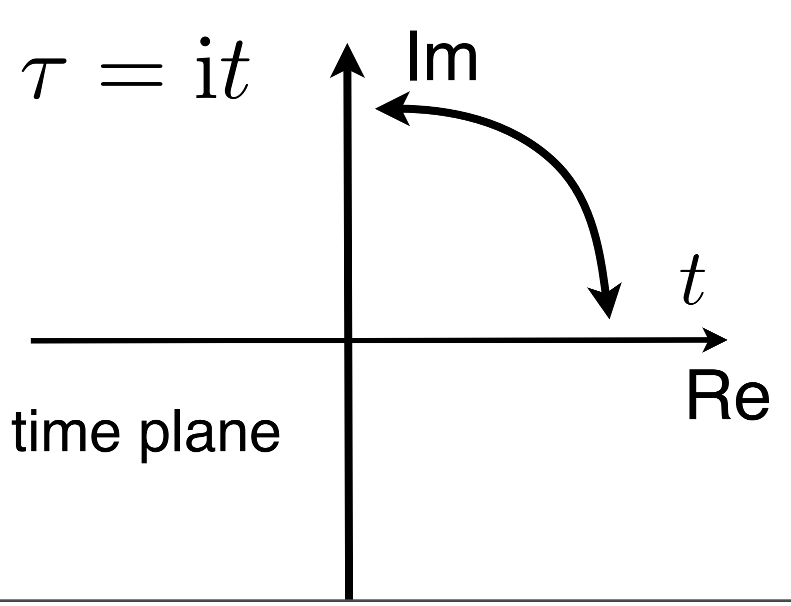
- Continue analytically the imaginary time classical action S to the real axis (at $T = 0$ or $\beta \to \infty$):

$$\begin{aligned} \tau \quad & \to \mathrm{i}t, \quad \phi(\tau, \mathbf{x}) \to \tilde{\phi}(t, \mathbf{x}) \Rightarrow \\ S[\phi^*, \phi] \quad & \to \mathrm{i}S[\tilde{\phi}^*, \tilde{\phi}] = \mathrm{i} \int dt \int d^3x \left[\tilde{\phi}^*(t, \mathbf{x})(-\mathrm{i}\partial_t - \tfrac{\triangle}{2M} - \mu)\tilde{\phi}(t, \mathbf{x}) + \tfrac{g}{2}(\tilde{\phi}^*(t, x)\tilde{\phi}(t, x))^2 \right] \end{aligned}$$

- The Gross-Pitaevski equation is the field equation for the real time classical action $\delta S / \delta \tilde{\phi}^*(t, \mathbf{x}) = 0$

$$\mathrm{i}\partial_t \tilde{\phi}(t, \mathbf{x}) = \big(- \tfrac{1}{2M} \triangle - \mu + g\tilde{\phi}^*(t, \mathbf{x})\tilde{\phi}(t, \mathbf{x}) \big) \tilde{\phi}(t, \mathbf{x})$$

- Remark: “classical” refers to the absence of fluctuations. Physically, the global phase coherence implied in this equations is a quantum mechanical effect, with observable consequences: cf. discussion of quantized vortices



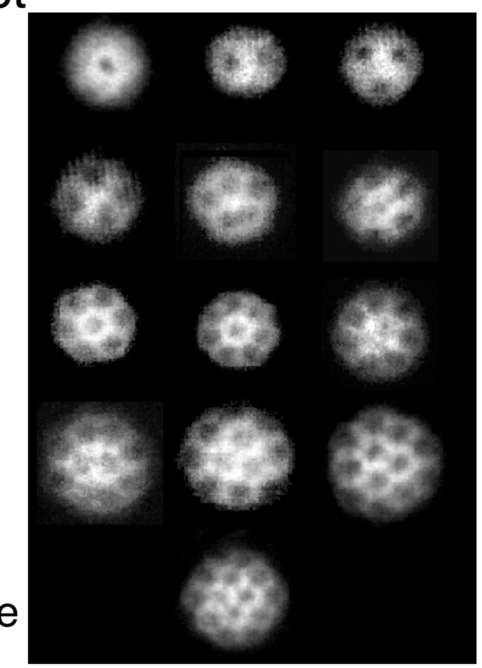
Complex time plane

Interpretation: Macroscopic Wave Function

- Gross-Pitaevski Equation (with trap):

$$\mathrm{i}\partial_t \varphi(\mathbf{x},t) = \Big(-\frac{\hbar^2}{2m}\triangle - \mu + V(\mathbf{x}) + g\varphi^*(\mathbf{x},t)\varphi(\mathbf{x},t)\Big)\varphi(\mathbf{x},t)$$

- Properties:
  - Classical field equation (cf. classical electrodynamics vs. QED)
  - for g = 0, or single particle: formally recover linear Schrödinger equation -> expect quantum behavior; interpret φ as “macroscopic wave function”
  - however, in general nonlinear -> richer than Schrödinger equation



- Interplay of quantum mechanics and nonlinearity: quantized vortex solutions
  - uniform case V(x) = 0, search static cylinder symmetric solutions with no z dependence:

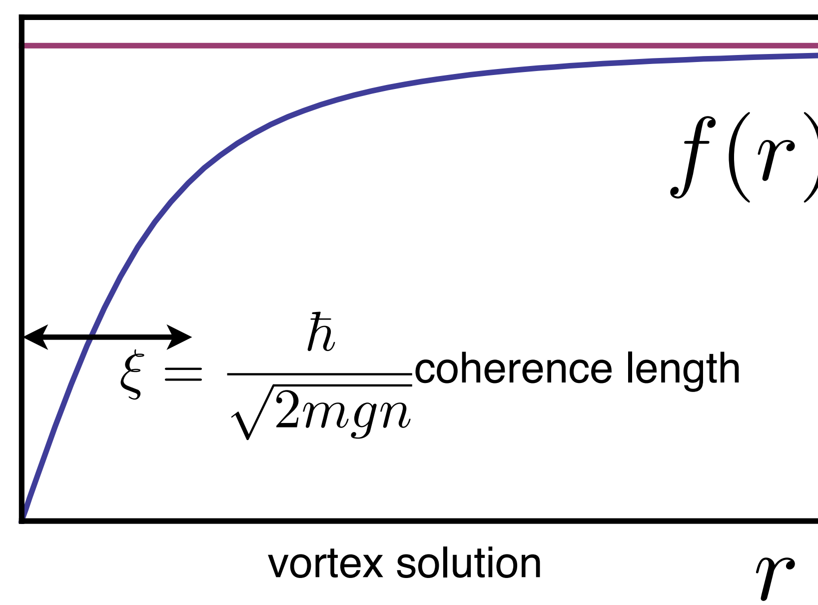
$$\varphi(\mathbf{x},t) = \varphi(r,\phi) = f(r)e^{\mathrm{i}\ell\phi}$$

integer, such that phase returns after 2 pi: unique Wave function

  - GP equation:

$$0 = -\frac{\hbar^2}{2m}\Big(f" + \frac{f'}{r} - \frac{\ell^2 f}{r^2}\Big) - \mu f + g f^3$$

  - large distances: constant solution, determine chemical pot.
  - short distances: condensate amplitude must vanish due to centrifugal barrier, in turn rooted in the quantization of the phase



vortex solution

Mean Field Action and Spontaneous Symmetry Breaking

- Specialize to homogeneous action: time- and space independent amplitudes ($\int d\tau d^3x = V/T$ -- quantization volume)

$$S(\phi^*, \phi) = V/T\underbrace{\left(-\mu\phi^*\phi + \tfrac{g}{2}(\phi^*\phi)^2\right)}_{\text{effective potential U}}$$

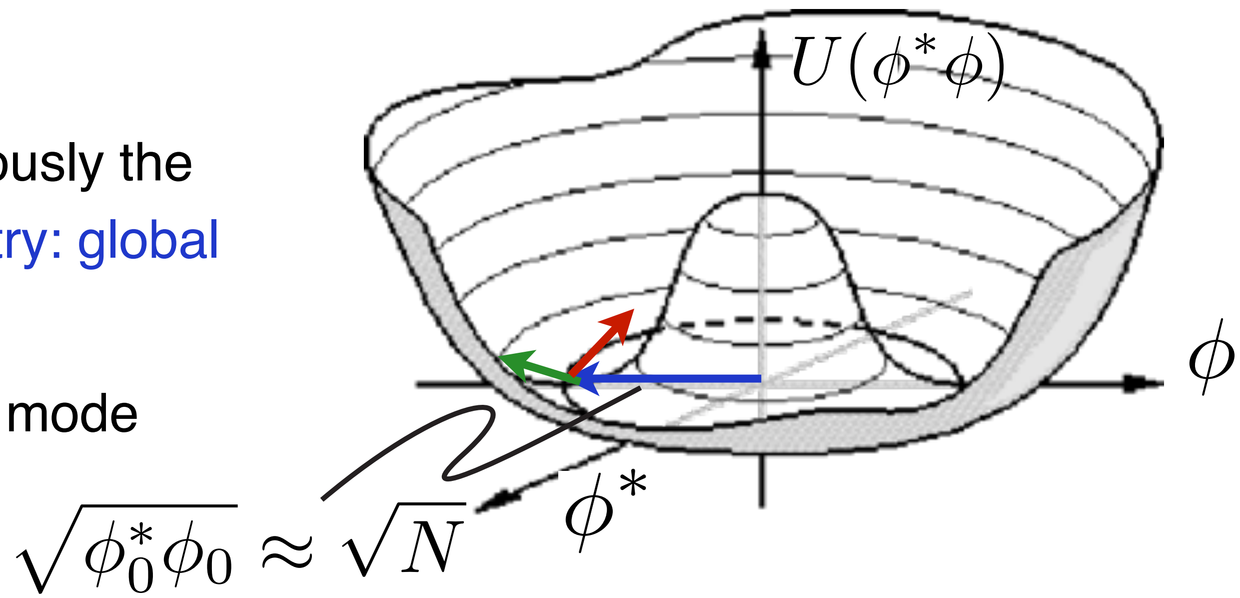
- homogeneous GPE or equilibrium condition for the classical field:

$$0 = \frac{\partial U}{\partial \phi^*} = \big(-\mu + g\phi^*\phi\big)\phi$$

- particle density:

$$n = -\frac{\partial U}{\partial \mu} = \phi^*\phi$$

- Geometrical interpretation: Mexican hat potential
  - for the ground state, the system chooses spontaneously the direction: spontaneous symmetry breaking (symmetry: global phase rotations U(1))
  - Radial (amplitude) excitations: cost energy, gapped mode
  - angular (phase) excitations: no energy cost due to degeneracy, gapless Goldstone mode



- The radial (amplitude) and angular (phase) excitations can be identified explicitly in the quadratic fluctuations (see below)

Quadratic Fluctuations: Bogoliubov Theory

- We go one step beyond the classical limit and include quadratic fluctuations on top of the mean field
- Expansion of S in powers of $(\delta\varphi^*, \delta\varphi)$ around $(\delta\varphi^*, \delta\varphi) = (0,0)$ yields the approximate effective action (saddle point approximation):

$$\begin{aligned}\Gamma[\phi^*,\phi] &= -\log \int \mathcal{D}(\delta\varphi^*,\delta\varphi) \exp -S[\phi^* + \delta\varphi^*, \phi + \delta\varphi] \\ &\approx S[\phi^*,\phi] - \log \int \mathcal{D}(\delta\varphi^*,\delta\varphi) \exp -\tfrac{1}{2}\int (\delta\varphi,\delta\varphi^*) S^{(2)}[\phi^*,\phi] \begin{pmatrix} \delta\varphi \\ \delta\varphi^* \end{pmatrix}\end{aligned}$$

Here, we have used the field equation $\delta S/\delta(\delta\varphi) = \delta S/\delta\phi = 0$

- We restrict to the homogeneous case $\phi(\tau,\mathbf{x}) = \phi_0$ for the condensate mean field. Then, the exponent reads in Fourier space ($Q = (\omega_n, \mathbf{q})$, $\int_Q = \sum_n T \int \frac{d^3q}{(2\pi)^3}$):

$$S_F = \frac{1}{2}\int_Q (\delta\varphi(-Q), \delta\varphi^*(Q)) \begin{pmatrix} g\phi_0^{*2} & -\mathrm{i}\omega_\mathrm{n} + \frac{\mathbf{q}^2}{2\mathrm{M}} - \mu + 2g\phi_0^*\phi_0 \\ \mathrm{i}\omega_\mathrm{n} + \frac{\mathbf{q}^2}{2\mathrm{M}} - \mu + 2g\phi_0^*\phi_0 & g\phi_0^2 \end{pmatrix} \begin{pmatrix} \delta\varphi(Q) \\ \delta\varphi^*(-Q) \end{pmatrix}$$

where we have to insert the solution of the homogeneous field equation $0 = \frac{\delta S}{\delta\phi}\Big|_{\mathrm{hom.}} = (-\mu + g\phi_0^*\phi_0)\phi_0^*$, i.e. $\mu = g\phi_0^*\phi_0$.

- NB: The remaining functional integral is Gaussian and can be done exactly. One can calculate rough estimates for e.g. the interaction induced density depletion at zero temperature from it.

The Excitation Spectrum

- The excitation spectrum / dispersion relation obtains from the poles of the propagator G, or the zeroes of $S^{(2)} = G^{-1}$ (analytically continued to real continuous frequencies $E = \mathrm{i}\omega_n$)

$$\det G^{-1}(\mathrm{E} = \mathrm{i}\omega, \mathbf{q}) \overset{!}{=} 0 \qquad \Rightarrow \boxed{E_{\mathbf{q}} = \sqrt{\epsilon_{\mathbf{q}}(\epsilon_{\mathbf{q}} + 2g\rho_0)}}$$

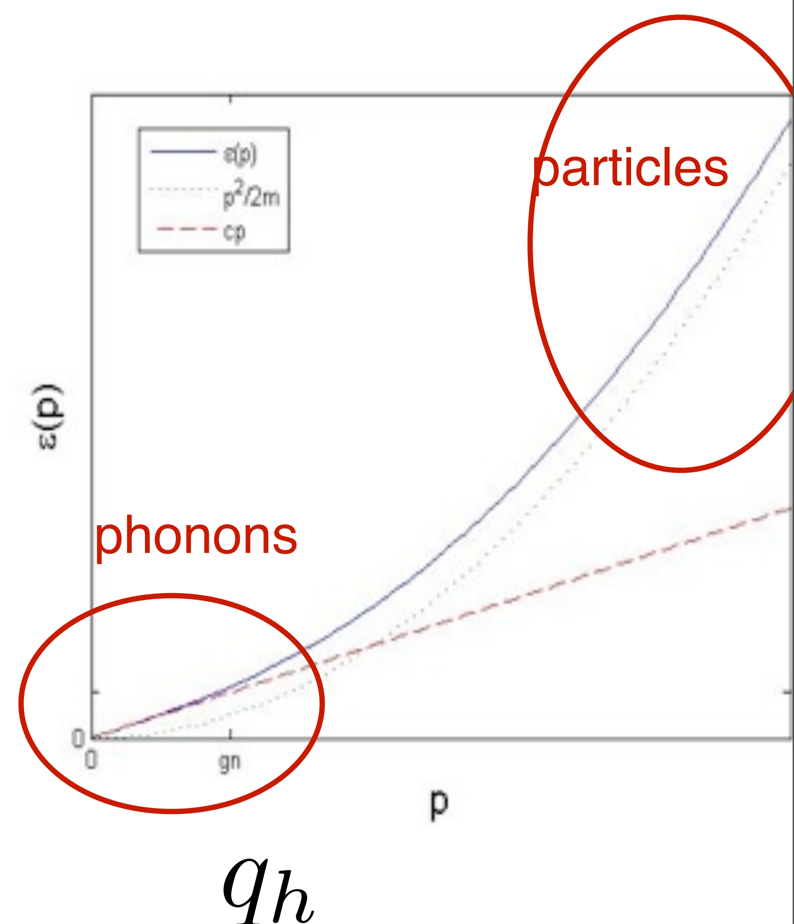
- Discussion:
  - At low momenta, this is linear and gapless, reminiscent of acoustic phonons or relativistic dispersions

$$E_{\mathbf{q}} \overset{\mathbf{q}\to 0}{\to} c|\mathbf{q}|, \qquad c = \sqrt{\frac{g\rho_0}{m}}$$

speed of sound

  - At high momenta, like free particles: quadratic

$$E_{\mathbf{q}} \overset{\mathbf{q}\to\infty}{\to} \epsilon_{\mathbf{q}} = \frac{\mathbf{q}^2}{2M}$$



- The regimes are separated by the “healing” momentum scale

$$q_h = \sqrt{2}mc = \sqrt{2gm\rho_0}$$

- Its inverse is the “healing length” $\xi_h = q_h^{-1}$, which is e.g. the characteristic size of a vortex, where the homogenous condensate “heals” (see above).

Phase and Amplitude Fluctuations

- We analyze the quadratic action for the boson fluctuations, using $-\mu = g\phi^*\phi$

$$S_F[\delta\varphi^*, \delta\varphi] = \frac{1}{2}\int_Q (\delta\varphi(-Q), \delta\varphi^*(Q)) \begin{pmatrix} g\phi_0^{*2} & -\mathrm{i}\omega_\mathrm{n} + \epsilon_\mathbf{q} + g\phi_0^*\phi_0 \\ \mathrm{i}\omega_\mathrm{n} + \epsilon_\mathbf{q} + g\phi_0^*\phi_0 & g\phi_0^2 \end{pmatrix} \begin{pmatrix} \delta\varphi(Q) \\ \delta\varphi^*(-Q) \end{pmatrix}$$

- We perform a change of basis (real and imaginary parts),

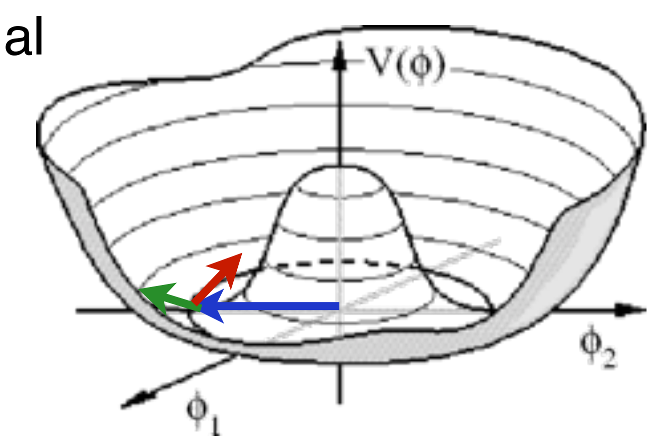
$$\delta\varphi_1(Q) = (\delta\varphi^*(-Q) + \delta\varphi(Q))/\sqrt{2}, \quad \delta\varphi_2(Q) = \mathrm{i}(\delta\varphi^*(Q) - \delta\varphi(-Q))/\sqrt{2}$$

- The action in the new coordinates reads ($\rho_0 = \phi_0^*\phi_0$ and we choose ϕ real without loss of generality)

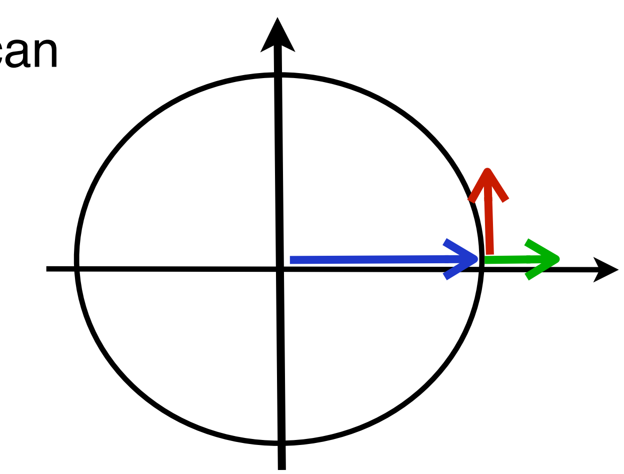
$$S_F[\delta\varphi_1, \delta\varphi_2] = \frac{1}{2}\int_Q (\delta\varphi_1(-Q), \delta\varphi_2(Q)) \begin{pmatrix} \epsilon_\mathbf{q} + 2g\rho_0 & -\omega_\mathrm{n} \\ \omega_\mathrm{n} & \epsilon_\mathbf{q} \end{pmatrix} \begin{pmatrix} \delta\varphi_1(Q) \\ \delta\varphi_2(-Q) \end{pmatrix}$$

  – Real part: amplitude fluctuations (see figure); these are gapped (massive) with $2g\rho_0$
  – Imaginary part: phase fluctuations; these are gapless (massless)

Mexican hat potential



Bottom of Mexican hat potential



➡ Origin of the phonon mode: fluctuations of the phase

➡ More generally, phonon mode is manifestation of Goldstone theorem in nonrel. system

Physical Significance: Phonon Mode and Superfluidity

- Landau criterion of superfluidity: frictionless flow
  - Gedankenexperiment: move an object through a liquid with velocity v.
  - Landau: the creation of an excitation with momentum p and energy $\epsilon_{\mathbf{p}}$ is energetically unfavorable if

$$\boxed{v < v_c = \frac{\epsilon_{\mathbf{p}}}{p}}$$

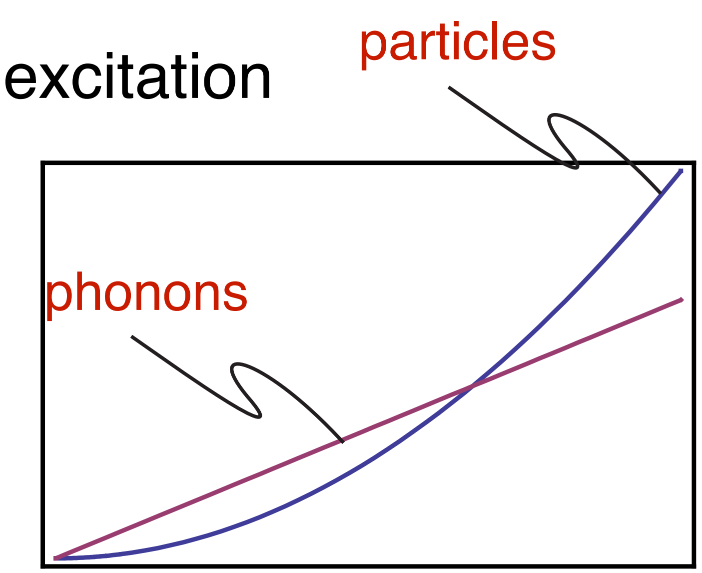
➡in this case, the flow is frictionless, i.e. superfluidity is present

- Weakly interacting Bose gas: Superfluidity through linear phonon excitation

$$\epsilon_{\mathbf{p}} = c|\mathbf{p}|, c = \sqrt{\frac{gn_0}{m}} \rightarrow v_c = c$$

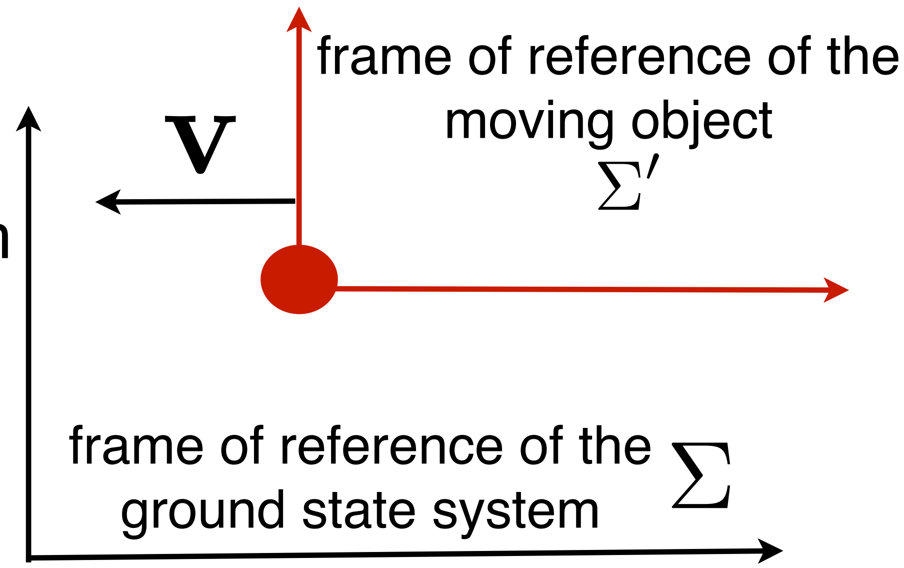
- Free Bose gas: No superfluidity due to soft particle excitations

$$\epsilon_{\mathbf{p}} = \frac{\mathbf{p}^2}{2m} \rightarrow v_c = 0$$



➡ Superfluidity is due to linear spectrum of quasiparticle excitations additional material

Idea of Landau Criterion



- Consider moving object in the liquid ground state of a system
- Question: When is it favorable to create excitations?

  - General transformation of energy and momentum under Galilean boost with velocity $\mathbf{v}$

$$\Sigma: \quad E, \quad \mathbf{p}$$

total system mass

$$\Sigma': \quad E' = E - \mathbf{p}\mathbf{v} + \tfrac{1}{2}M\mathbf{v}^2, \quad \mathbf{p}' = \mathbf{p} - M\mathbf{v}$$

  - Energy and momentum of the ground state in

$$\Sigma: \quad E_0, \quad \mathbf{p}_0 = 0$$

$$\Sigma': \quad E_0' = E_0 + \tfrac{1}{2}M\mathbf{v}^2, \quad \mathbf{p}_0' = -M\mathbf{v}$$

  - Energy and momentum of the ground state plus an excitation with momentum, energy $\mathbf{p}, \epsilon_{\mathbf{p}}$

$$\Sigma: \quad E_{\text{ex}} = E_0 + \epsilon_{\mathbf{p}}, \quad \mathbf{p}_{\text{ex}} = \mathbf{p}$$

$$\Sigma': \quad E_{\text{ex}}' = E_0 + \epsilon_{\mathbf{p}} - \mathbf{p}\mathbf{v} + \tfrac{1}{2}M\mathbf{v}^2, \quad \mathbf{p}_{\text{ex}}' = \mathbf{p} - M\mathbf{v}$$

  - Creation of excitation unfavorable if

$$E_{\text{ex}}' - E_0' = \epsilon_{\mathbf{p}} - \mathbf{p}\mathbf{v} \geq \epsilon_{\mathbf{p}} - |\mathbf{p}||\mathbf{v}| > 0 \qquad \Rightarrow v < v_{\text{c}} = \frac{\epsilon_{\mathbf{p}}}{p}$$

Validity of Bogoliubov Theory

- The ordering principle of the semiclassical approximation is the existence of a macroscopic (extensive) condensate, i.e.:

$$\exp - \frac{\Gamma[\phi^*,\phi]}{\hbar} = \int \mathcal{D}(\delta\varphi^*,\delta\varphi) \exp - \frac{S[\phi^*+\delta\varphi^*,\phi+\delta\varphi]}{\hbar}$$

$$\phi \sim N^{1/2} \sim V^{1/2}, \quad \delta\varphi \sim N^0$$

the ordering principle is not $\hbar \to 0$

- Obviously, Bogoliubov theory breaks down if no condensate exists. This situation appears for

$$d = 1 \quad \text{all } T$$

$$d = 2, \quad T > 0$$

Mermin-Wagner theorem plus dimensional reduction

- In these cases, immediate need for nonperturbative approaches such as (functional) RG (Castellani& '04, Wetterich & '08,'09; Kopietz & '08,'10; Dupuis '09)

- even in d=3, or d=2,T=0, one should be suspicious since in the range of small momenta the power counting is questionable:

$$n_{\mathbf{q}} = \int_\omega \langle \delta\varphi^*_Q \delta\varphi_Q \rangle \sim 1/E_{\mathbf{q}} \sim 1/|\mathbf{q}|$$

divergent occupation number

$$\to \delta\varphi \sim 1/|\mathbf{q}|^{1/2} \quad ?!$$

Validity of Bogoliubov Theory

- We study perturbative corrections to the self-energy for weakly interacting bosons (zero temperature): The full quadratic part of the effective action is ($Q = (\omega, \mathbf{q})$)

$$\Gamma = \frac{1}{2}\int_Q (\varphi(-Q), \varphi^*(Q)) \begin{pmatrix} \Sigma_{an}(Q) & -\mathrm{i}\omega + \frac{\mathbf{q}^2}{2\mathrm{M}} - \mu + \Sigma_n(Q) \\ \mathrm{i}\omega + \frac{\mathbf{q}^2}{2\mathrm{M}} - \mu + \Sigma_n(Q) & \Sigma_{an}(Q) \end{pmatrix} \begin{pmatrix} \varphi(Q) \\ \varphi^*(-Q) \end{pmatrix}$$

- We may view Bogoliubov Theory as the zero order self energies

$$\Sigma_n^{(0)}(Q) = 2g\rho_0, \quad \Sigma_{an}^{(0)}(Q) = g\rho_0 \qquad\qquad \rho_0 = \phi_0^*\phi_0$$

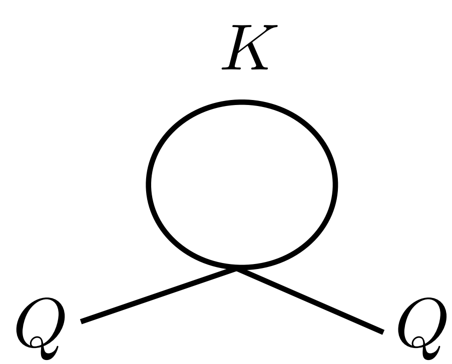
- The leading pertrubative corrections are shown diagrammatically. The second diagram has in IR divergence (log in d=3, poly in d<3)

$$\Sigma_n^{(1)}(Q) \sim \Sigma_{an}^{(1)}(Q) \sim -g^2\rho_0 \int_K G_{22}(K)G_{22}(Q+K), \quad G_{22}(Q) = \frac{2g\rho_0}{\omega^2 + c^2\mathbf{q}^2}$$

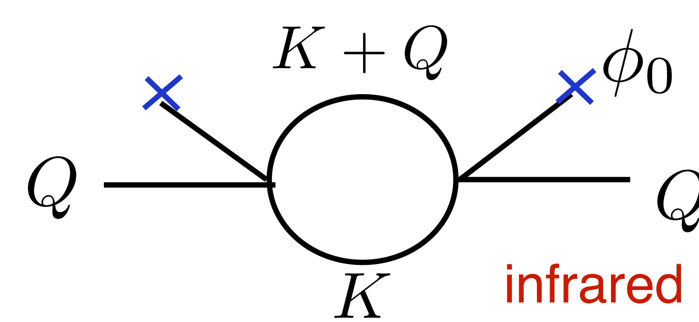
- Perturbation theory breaks down for

$$|\Sigma_{n,an}^{(0)}(Q)| \approx |\Sigma_{n,an}^{(1)}(Q)|$$

first order perturbation theory



infrared regular



infrared divergent!

Weakly and Strongly Correlated Superfluid

- Perturbation theory breaks down for

$$|\Sigma^{(0)}_{n,an}(Q)| \approx |\Sigma^{(1)}_{n,an}(Q)|$$

from which we deduce the scale where the superfluid becomes nonperturbative/strongly correlated

$$p_{np} = p_h \begin{cases} \tilde{g}^{\frac{d}{2(3-d)}} & \text{if} \quad d < 3 \\ \exp(-\frac{1}{\kappa \tilde{g}^{3/2}}) & \text{if} \quad d = 3 \end{cases}$$

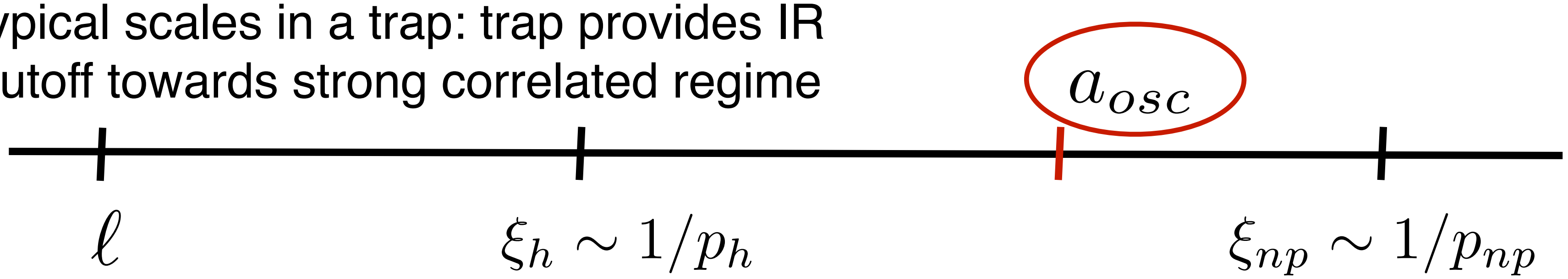
- The dimensionless ratio $\tilde{g}$ expresses the ratio of interaction versus kinetic energy in the nonrelativistic superfluid ($[g] = 2 - d$):

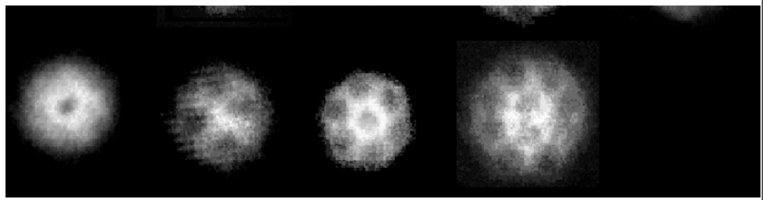
$$\tilde{g} = \frac{E_{pot}}{E_{kin}} = \frac{g\rho_0}{1/(M\ell^2)} = gM\rho_0^{1-2/d} \sim (p_h \ell)^2$$

where $\ell \sim n^{-1/d}$ is the mean interparticle distance and $n \approx \rho_0$ in the weakly interacting condensate

- Thus, superfluids can be classified according to:
  - weakly correlated if $\tilde{g} \ll 1 \Rightarrow p_{np} \ll p_h \ll \ell^{-1}$. Bogoliubov theory is valid for a large part of the spectrum, namely for momenta $|\mathbf{q}| \gtrsim p_{np}$. This is the case in typical traps.
  - strongly correlated if $\tilde{g} \gtrsim 1 \Rightarrow p_{np} \approx p_h \approx \ell^{-1}$. Bogoliubov theory breaks down. This may happen on the lattice close to the Mott insulator – superfluid phase transition.

typical scales in a trap: trap provides IR cutoff towards strong correlated regime





visualization of the scales

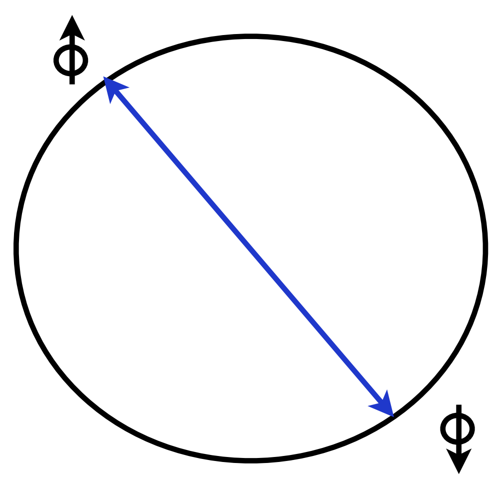
ξ_h -- vortex size

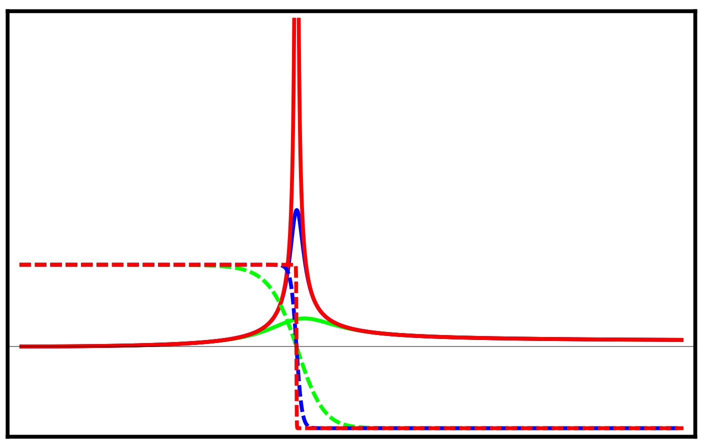
a_{osc} -- extent of cloud

Weakly Interacting Fermions

$$\langle \psi_\uparrow \rangle = \langle \psi_\downarrow \rangle = 0$$

$$\langle \psi_\uparrow \psi_\downarrow \rangle \neq 0$$





Free Fermions and Fermi Momentum

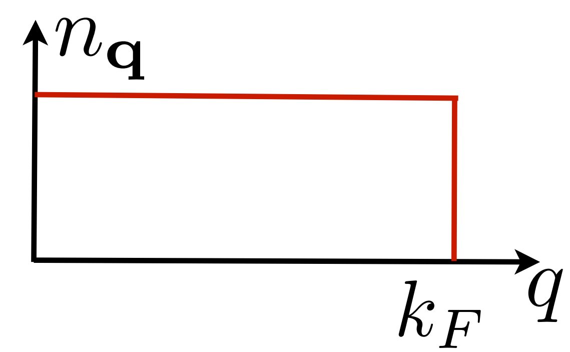
- Collection of some useful formulae and abbreviations for 3D two-component fermions:
- The equation of state for free fermions at zero temperature:

$$n = 2\int \frac{d^3q}{(2\pi)^3}(\exp(\tfrac{\epsilon_{\mathbf{q}}-\mu}{T}+1)^{-1} \xrightarrow{T\to 0} 2\int\frac{d^3q}{(2\pi)^3}\theta(\epsilon_{\mathbf{q}}-\mu) = \frac{(2M\mu)^{3/2}}{3\pi^2} \equiv \frac{k_{\mathrm{F}}^3}{3\pi^2}$$

two spin states

- The Fermi momentum k_F is defined as the momentum scale associated to the chemical potential of free fermions at T = 0

$$k_{\mathrm{F}} \equiv (2M\mu^{(\mathrm{free})}_{T=0})^{1/2}$$



- It is a measure for the total density of a fermion system, and therefore for the mean interparticle spacing:

$$d = (3\pi^2)^{1/3}k_{\mathrm{F}}^{-1}$$

- The associated energy and temperature scales are the Fermi energy and the Fermi temperature

$$\epsilon_{\mathrm{F}} = \frac{k_{\mathrm{F}}^2}{2M}, \quad T_{\mathrm{F}} = \frac{\epsilon_{\mathrm{F}}}{k_{\mathrm{B}}}$$

Physical Picture for Weakly Attractive Fermions

- The low temperature physics of free fermions is governed by the Pauli principle

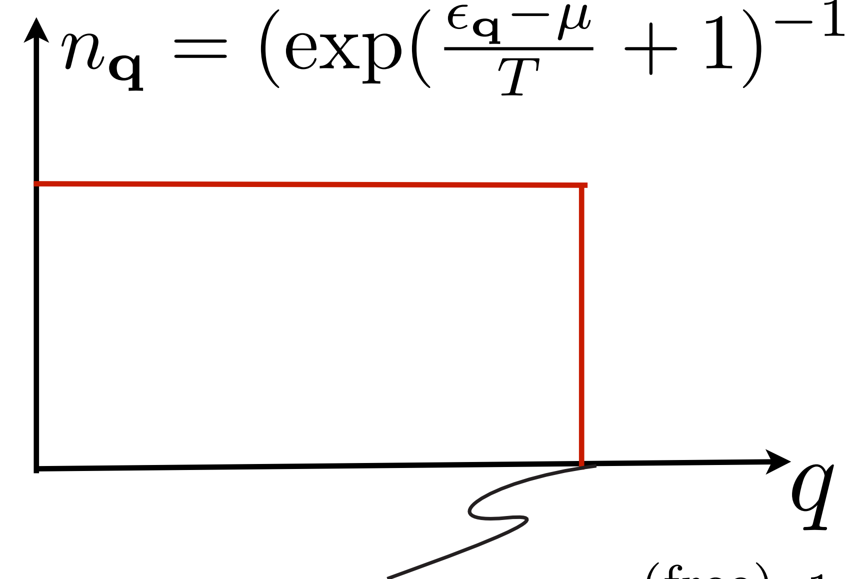
  (1) Expression of a Fermi sphere in momentum space

  (2) Absence of fermion condensation: $\langle\psi_\sigma\rangle = 0 \quad \sigma = \uparrow, \downarrow$

  (3) Local s-wave interactions of fermions are only possible for more than one spin state (ultracold atoms: hyperfine states)

Fermi distribution at low T

$$n_{\mathbf{q}} = (\exp(\tfrac{\epsilon_{\mathbf{q}} - \mu}{T}) + 1)^{-1}$$



Fermi momentum $k_{\mathrm{F}} \equiv (2M\mu^{(\mathrm{free})}_{T=0})^{1/2}$

- Now we allow for weak 2-body s-wave attraction between 2 spin states of fermions

$$a < 0 \qquad\qquad |ak_{\mathrm{F}}| \sim |a/d| \ll 1$$

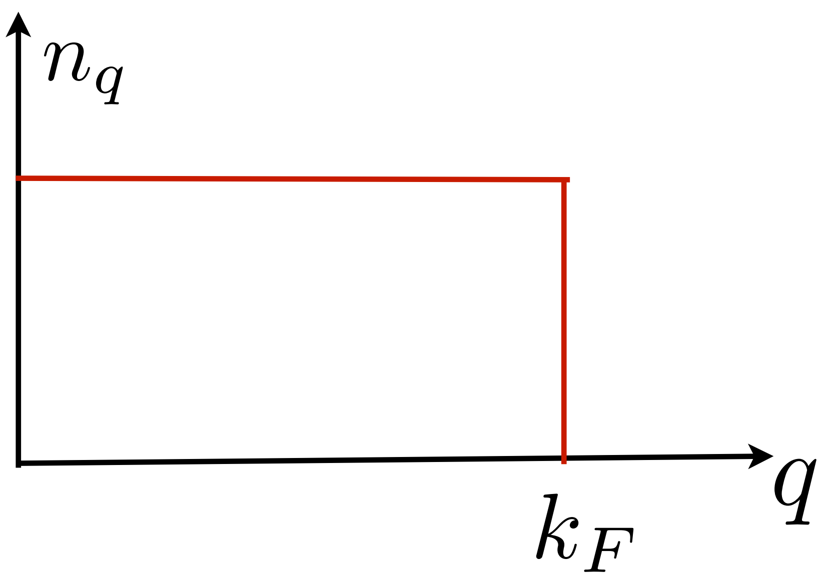
attractive scattering length — weakness/diluteness condition

- A small interaction scale will not be able to substantially modify the Fermi sphere. This is the key to BCS theory

Physical Picture for Weakly Attractive Fermions

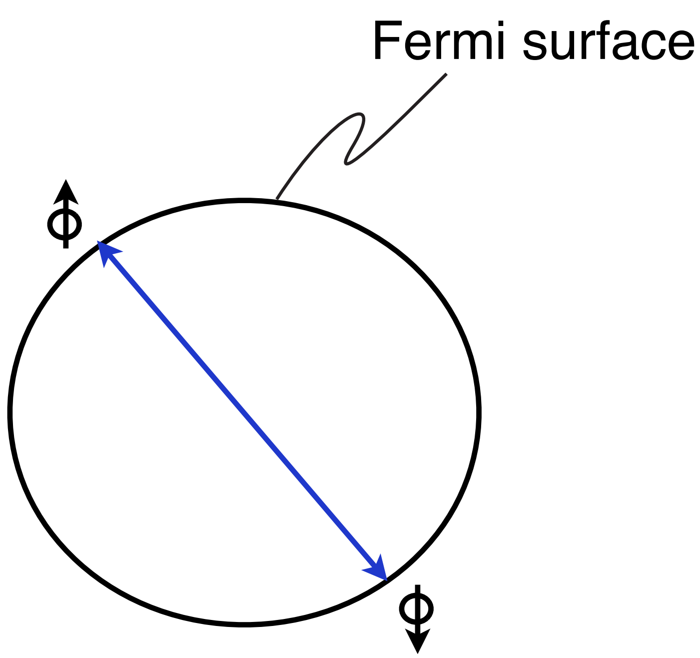
$$|ak_{\mathrm{F}}| \sim |a/d| \ll 1$$

Fermi distribution at low T



- A small interaction scale will not be able to substantially modify the Fermi sphere
- However, pairing of fermions with momenta close to the Fermi surface is possible: "Cooper pairs":
  - These fermions attract each other with strength a
  - The total energy of the system is lowered when
    - bosonic pairs with zero cm energy (total momentum zero) form: local in momentum space
    - These pairs condense, i.e. occupy a single quantum state macroscopically:

$$\boxed{\langle \psi_\uparrow \psi_\downarrow \rangle \neq 0}$$



Cooper pairing: Local in momentum space

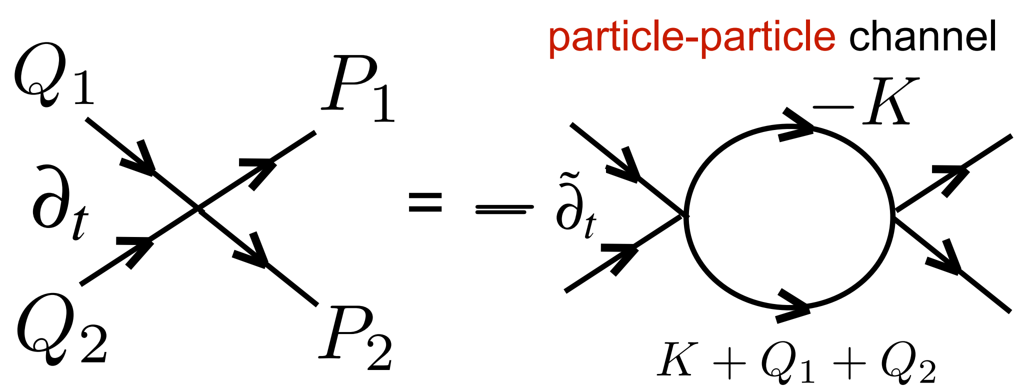
- Comments:
  - Distinguish pairing correlation from Bose condensation $\langle \varphi \rangle \neq 0$
  - But: in both cases, spontaneous breaking of U(1) symmetry

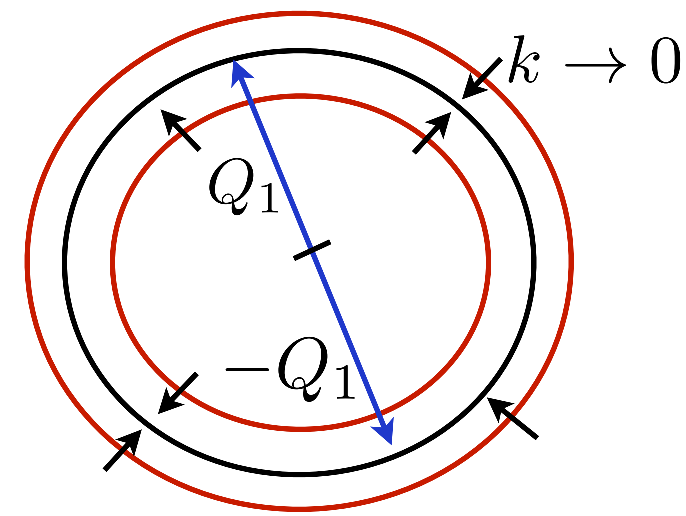
RG Argument for BCS Instability

$\psi = (\psi_\uparrow, \psi_\downarrow)^T$ two-component spinor

- Purely fermionic description $$S[\psi] = \int d\tau \int d^3x \Big\{ \psi^\dagger (\partial_\tau - \frac{\triangle}{2M} - \mu)\psi + \frac{\lambda}{2}(\psi^\dagger\psi)^2 \Big\}$$

- RG Equation with dominant particle-particle loop:





- Choose cutoff that approaches Fermi surface (FS) = IR limit for fermions shell by shell as displayed

- Study flow of the vertex $\lambda(Q_1, Q_2; P_1, P_2)$

see R. Shankar '93, "Renormalization Group approach to interacting Fermions"

  - Since the coupling is small, generically the renormalization effects are perturbatively small

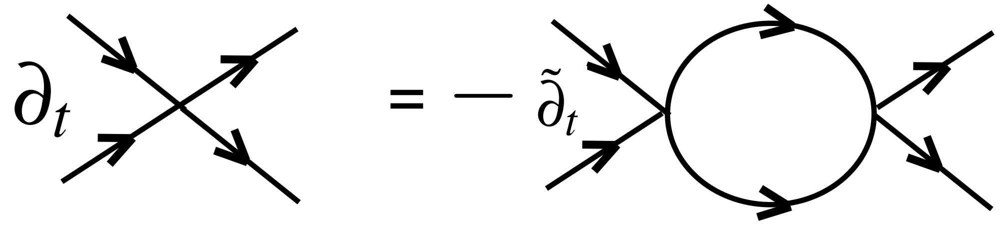
  - But the one with opposite spatial momenta and energies on the FS renormalizes strongly: The integral (zero temperature) is logarithmically divergent for $k \to 0$

$$\lambda \equiv \lambda(Q_1 = -Q_2; P_1 = -P_2)$$ spatial momenta opposite

  - The divergence drives the system to strong coupling for attractive interactions $\lambda_{\text{in}} < 0$ A physical instability against pairing occurs

BCS Instability

- Restricting to the single strongly flowing coupling on the FS, we have a simple quadratic beta-function



$$\partial_k \lambda_k = -\lambda_k^2 \tilde{\partial}_k I_k(T,\mu)$$

- Solution for $k \to 0$

$$\lambda_0 = \frac{1}{\lambda_{\rm in}^{-1} + I_0(T,\mu))}$$

- a finite temperature acts as physical IR cutoff. For low temperatures,

$$I_0(T \to 0, \mu > 0) \sim -\log T/\mu > 0$$

- Thus, for arbitrarily attractive interaction, a critical temperature exists where the interaction diverges.
- A more detailed analysis, including a proper UV Renormalization, yields (d=3, a the scattering length)

$$0 = -\frac{1}{a} - \frac{2}{\pi}\int dq \left[\frac{q^2}{q^2 - 2M\mu}\tanh\left(\frac{q^2/(2M) - \mu)}{2T}\right) - 1\right]$$

- The resulting critical temperature is

$$\boxed{\frac{T_{\rm c}}{\epsilon_{\rm F}} = \frac{8\gamma}{\pi e^2}\, e^{-\frac{\pi}{2|ak_{\rm F}|}}}$$

Euler constant $\gamma \approx 1.78$

prefactor ≈ 0.61

Experimental (Ir)relevance of Weakly Interacting Atomic Fermions

- We compare the critical temperatures for a noninteracting BEC and weakly attractive fermions
  - Free bosons of mass M undergo condensation at $n\lambda_{dB} = \zeta(3/2), \quad \lambda_{dB} = (2\pi/(MT))^{1/2}$
  - Rewrite by using definitions from fermions $n = k_{\mathrm{F}}^3/(3\pi^2), \quad \epsilon_{\mathrm{F}} = k_{\mathrm{F}}^2/(2M)$

$$\frac{T_{\mathrm{c}}^{(\mathrm{BEC})}}{\epsilon_{\mathrm{F}}} = 4\pi(3\pi^2\zeta(3/2))^{-2/3} \approx 0.69 = \mathcal{O}(1)$$

  - In contrast, the BCS critical temperature is exponentially small for $ak_{\mathrm{F}} \ll 1$

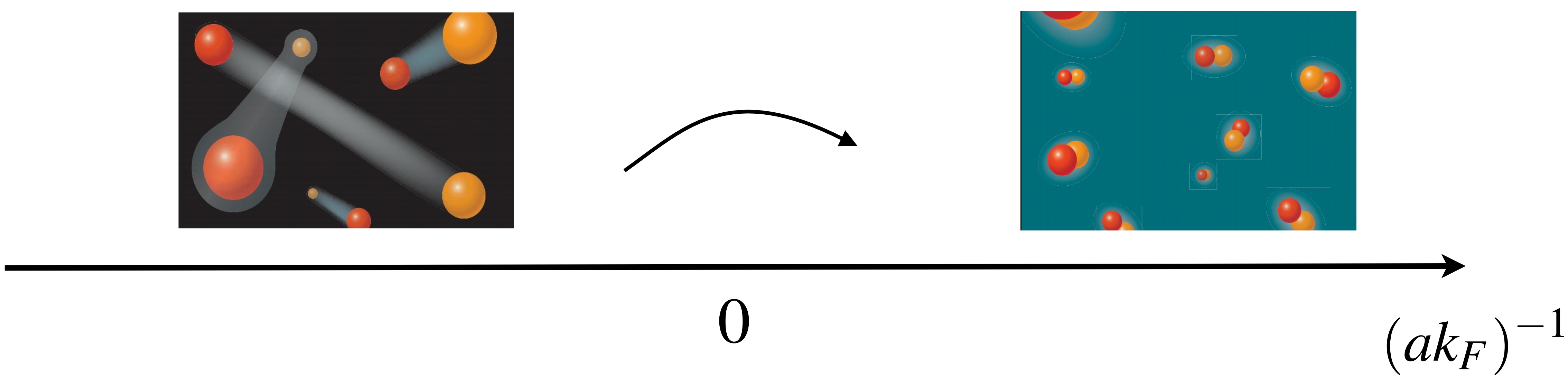
$$\frac{T_{\mathrm{c}}^{(\mathrm{BCS})}}{\epsilon_{\mathrm{F}}} = \frac{8\gamma}{\pi e^2}\, e^{-\frac{\pi}{2|ak_{\mathrm{F}}|}} \approx 0.61 e^{-\frac{\pi}{2|ak_{\mathrm{F}}|}}$$

- additionally, cooling of degenerate fermions is experimentally more challenging due to Pauli blocking

---

- On the other hand, note a (formal) exponential increase of T_c for rising ak_{F} i.e. towards strong interactions
- Q: What is the fate of the exponential increase in T_c for rising ak_{F}
- A: BCS-BEC crossover

Strong Interactions
and
the BCS-BEC Crossover

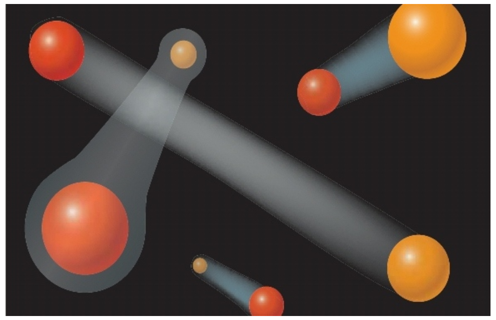


Physical picture: BCS-BEC Crossover

- We have discussed two cornerstones for quantum condensation phenomena:

  - fermions with attractive interactions

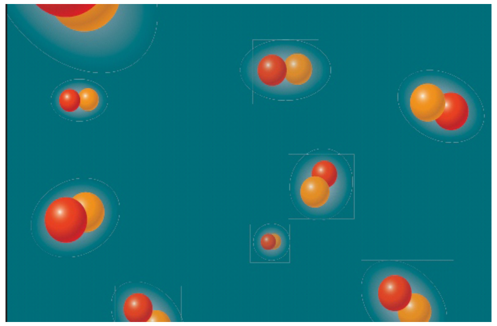
    ➡ BCS superfluidity at low T



  - weakly interacting bosons

    ➡ Bose-Einstein Condensate (BEC) at low T

bosons could be realized as
tightly bound molecules
("effective theory")



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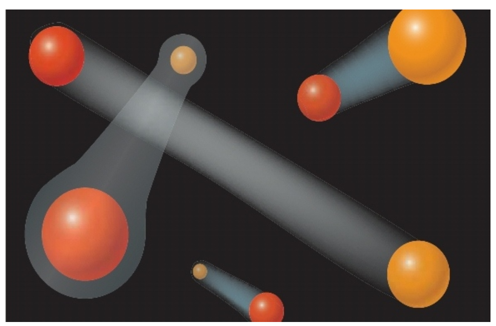
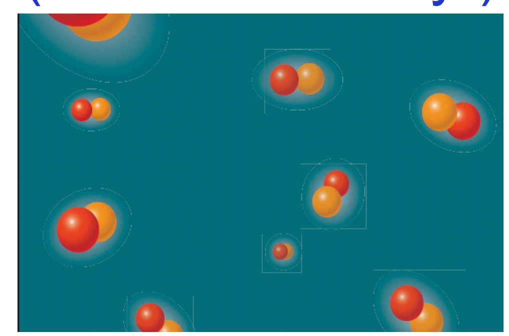




0 $\qquad (ak_F)^{-1}$

- There is an experimental knob to connect these scenarios: Feshbach resonances

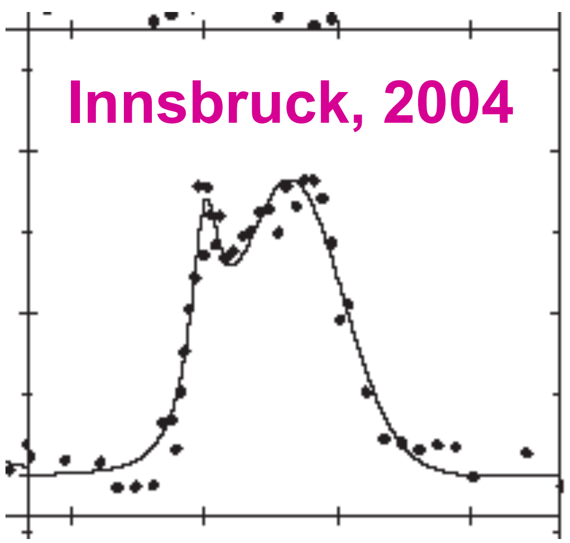
  - microscopically, the phenomenon is due to a bound state formation at the resonance $\frac{1}{ak_{\mathrm{F}}} = 0$
  - from a many-body perspective, the phenomenon is understood as

    - Localization in position space
    - Delocalization in momentum space

➡ In the strongly interacting regime, no simple ordering principle is known:
Challenge for Many-Body methods

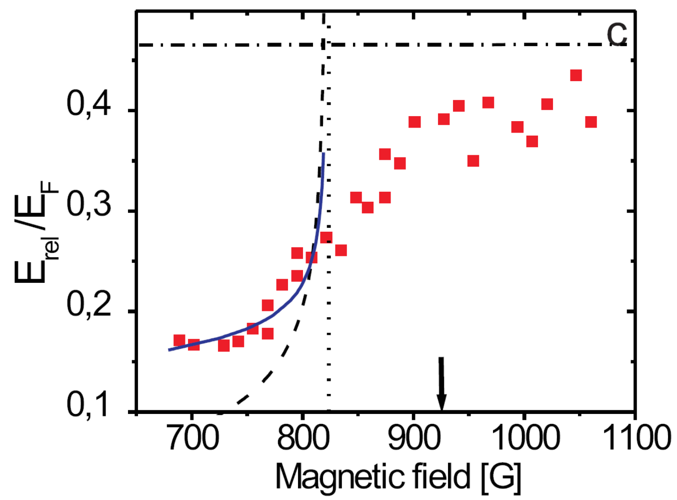
Experiments in the BCS-BEC Crossover

Innsbruck, 2004



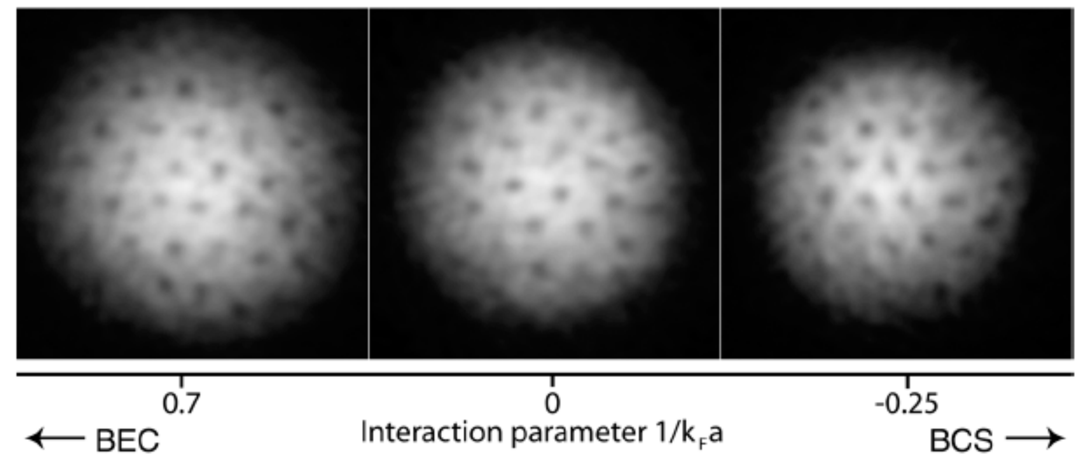
pairing gap

ENS, 2004

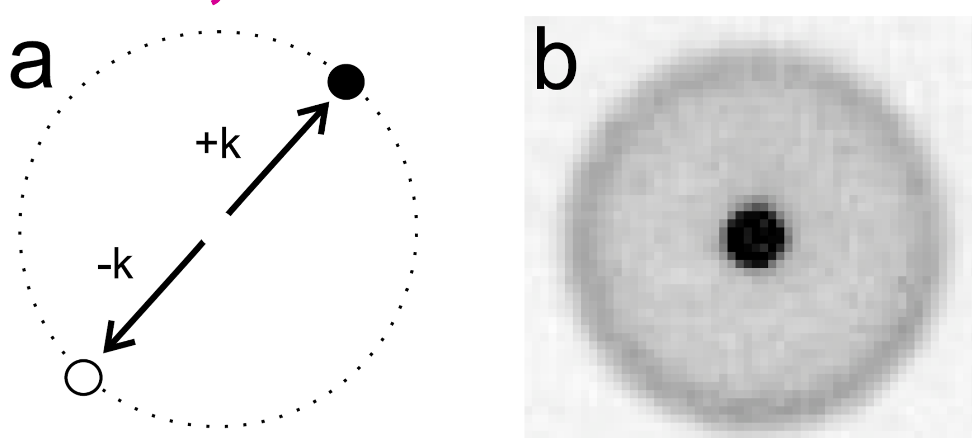


release energy

MIT, 2005 *vortices*

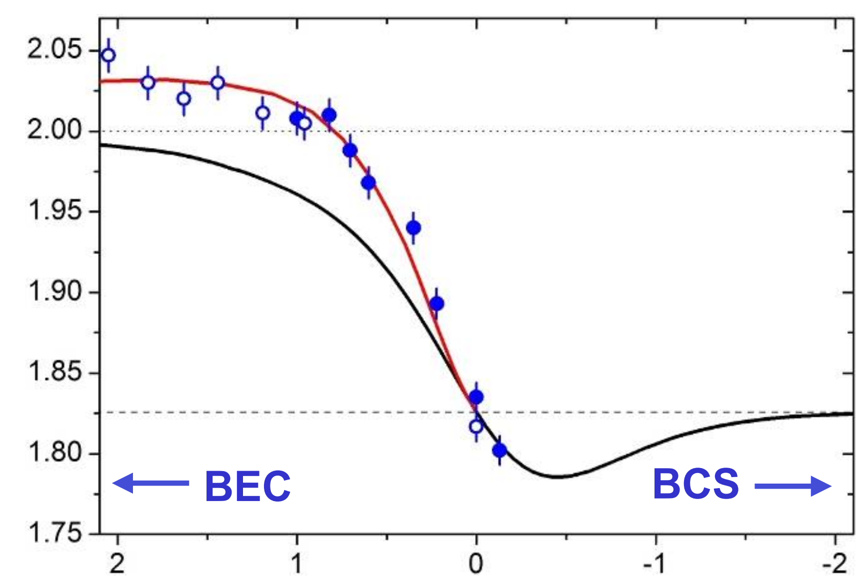


JILA, 2005



pair correlations

Duke ´04, Innsbruck 2004 & `06



collective modes

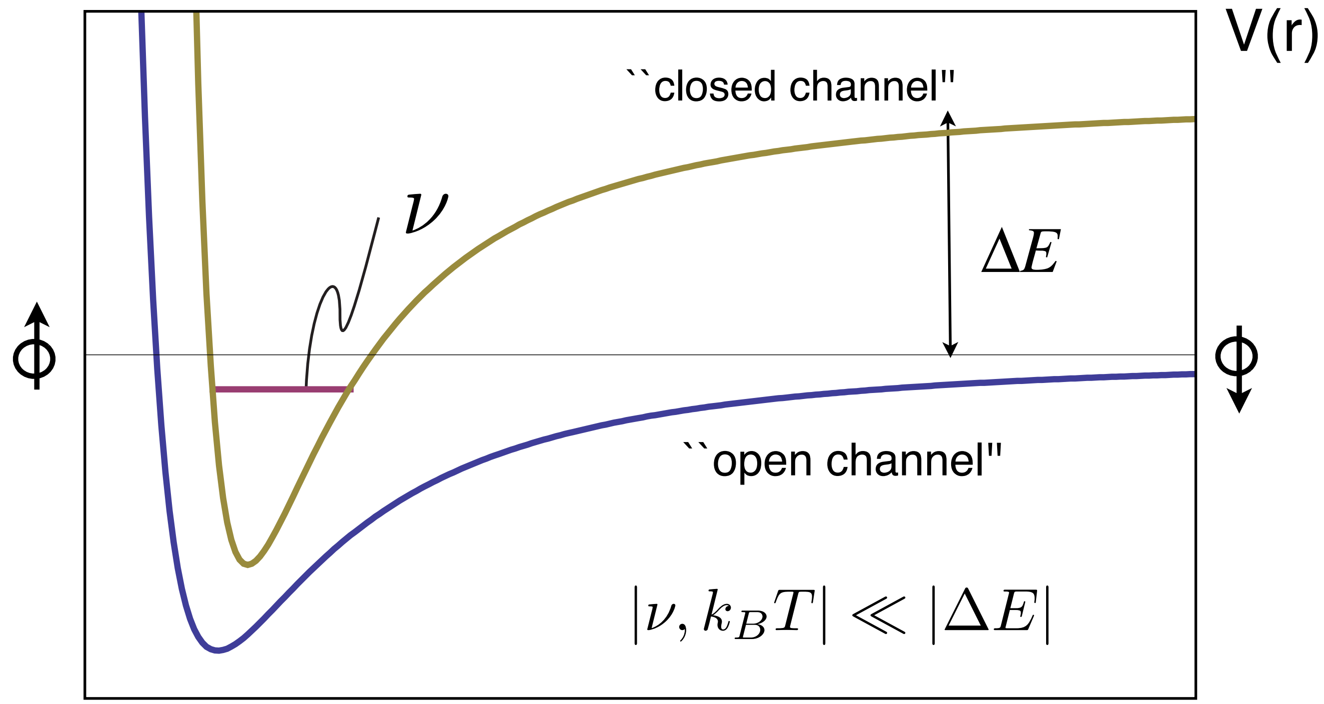
Microscopic Origin: Feshbach Resonances

- Start from fermions: (Euclidean) Action

fermion field: two hyperfine states $\psi = \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}$

$$S_\psi[\psi] = \int d\tau d^3x \big(\psi^\dagger(\partial_\tau - \tfrac{\triangle}{2M})\psi + \tfrac{\lambda_\psi}{2}(\psi^\dagger\psi)^2\big)$$

- Consider a second interaction channel with bound state close to scattering threshold V=0, detuned by ν



examples: $^6\mathrm{Li}, ^{40}\mathrm{K}$

- Detuning ν can be controlled with magnetic field

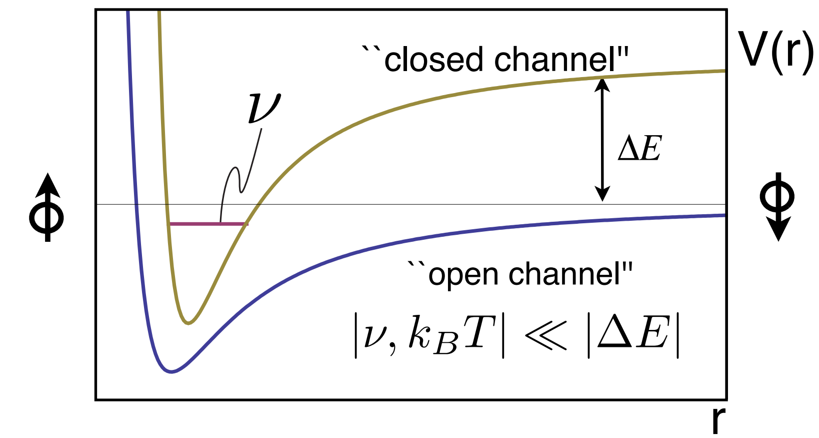
$$\nu(B) = \mu_B(B - B_0)$$

magnetic moment

resonance position

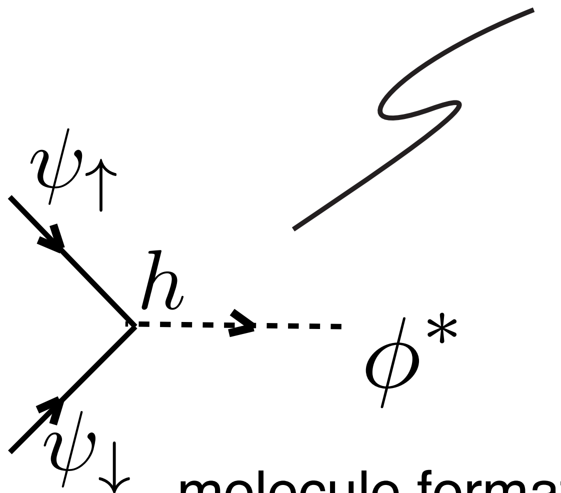
Microscopic Origin: Feshbach Resonances

- Effective Model to describe this situation:
  Interconversion of two fermions into a molecule



bosonic molecule field: $$S_\phi[\phi] = \int d\tau d^d x \phi^* (\partial_\tau - \frac{\triangle}{4M} + \nu)\phi$$

interconversion:
Feshbach, Yukawa term $$S_F[\psi, \phi] = -h \int d\tau d^d x (\phi^* \psi_\uparrow \psi_\downarrow - \phi \psi_\uparrow^* \psi_\downarrow^*)$$



molecule formation

- NB: cf. BCS Cooper pairing with condensate amplitude:

$$\phi_0^* = \text{const.}$$

- Now we allow for dynamic bosonic degrees of freedom

$$\phi^*(\tau, \mathbf{x}) \quad \text{or} \quad \phi^*(\omega, \mathbf{q})$$

- Parameters:
  - (background scattering in open channel) λ_ψ
  - Feshbach coupling: width of resonance h
  - detuning: distance from resonance ν

Relation to a strongly interacting theory

- Total action: $$S = S_\psi + S_\phi + S_F[\psi, \phi]$$

  Feshbach action

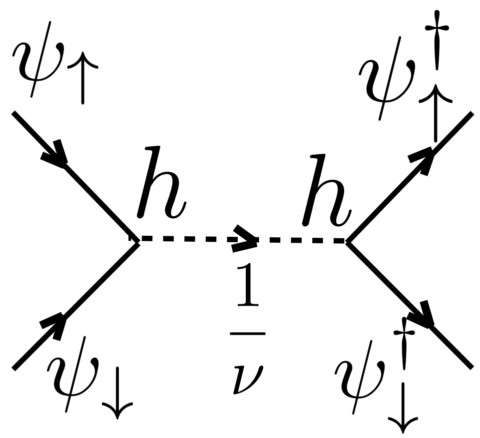
- Field equations:

$$\frac{\delta S}{\delta \phi^*} = 0 \qquad \Rightarrow (\partial_t - \tfrac{\triangle}{4M} + \nu)\phi = h\psi_\uparrow\psi_\downarrow$$

$$\Rightarrow \phi = \frac{h}{\partial_t - \frac{\triangle}{4M} + \nu}\psi_\uparrow\psi_\downarrow$$

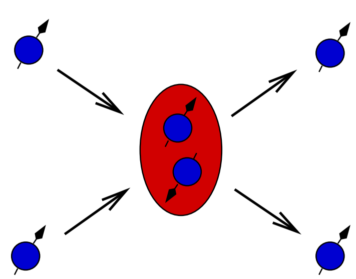
- Formally solve for ϕ, ϕ^*, and insert solution into the Feshbach term

$$S = S_\psi + \int d\tau d^3x \, \psi_\uparrow\psi_\downarrow \frac{h^2}{\cancel{\partial_t - \frac{\triangle}{2M}} + \nu}\psi_\uparrow^\dagger\psi_\downarrow^\dagger$$



- take constrained "broad resonance" limit:
  pointlike interactions

$$h \to \infty, \qquad \frac{h^2}{\nu} \to \text{const.}$$

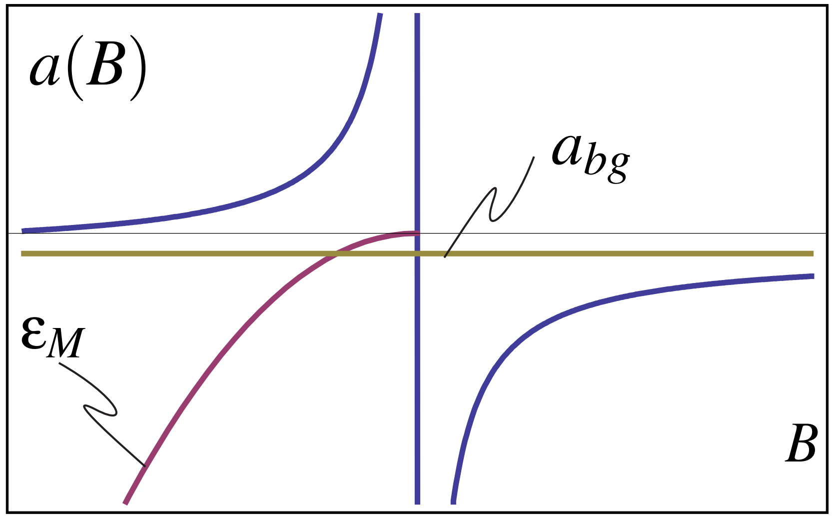


$$S \to S_\psi + \frac{h^2}{\nu}\int d\tau d^3x \, \psi_\uparrow^\dagger\psi_\downarrow^\dagger\psi_\uparrow\psi_\downarrow = S_\psi - \frac{1}{2}\frac{h^2}{\nu}\int d\tau d^3x \, (\psi^\dagger\psi)^2$$

Relation to a strongly interacting theory

- pointlike/broad resonance limit: The action takes the form

$$S[\psi] = \int d\tau d^d x \left(\psi^\dagger (\partial_t - \frac{\triangle}{2M})\psi + \frac{\lambda_\psi^{\text{eff}}}{2}(\psi^\dagger\psi)^2\right)$$



scattering length a and binding energy

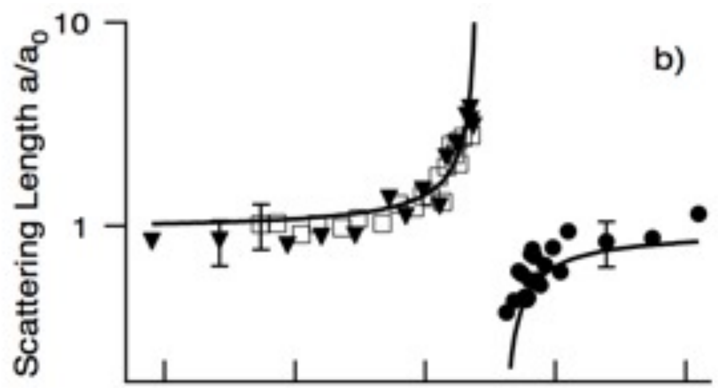
$$\lambda_\psi^{\text{eff}} = \lambda_\psi - \frac{h^2}{\nu(B)}$$

effective fermionic interaction

$$a = \frac{4\pi\lambda_\psi^{\text{eff}}}{M}$$

scattering length (nonidentical fermions)

observation of divergent scattering length
Ketterle Group, MIT (1999)
bosonic sodium



- remember $\nu(B) = \mu_B(B - B_0)$

  ➡ resonant (divergent) interaction at B_0

- in the following, we shall work in broad resonance limit and ignore the background scattering for simplicity

Regimes in the BCS-BEC Crossover

- Compare the scattering length to the mean interparticle spacing $d = (3\pi^2 n)^{-1/3}$

➡ three regimes

$a < 0,\ |a/d| \ll 1$ weakly interacting (dilute) fermions

$|a/d| \gtrsim 1$ strong interactions, dense

$a > 0,\ |a/d| \ll 1$ molecular bound states: dilute bosons
➡ see below!

- We identify the inverse scattering length as an adequate "crossover parameter"

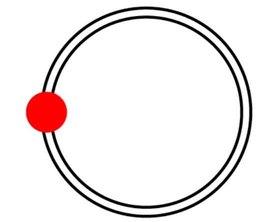
$$a^{-1}(B) = -\frac{M\nu(B)}{4\pi h^2}$$

since the Feshbach resonance is located at the zero crossing of the detuning $\nu(B)$

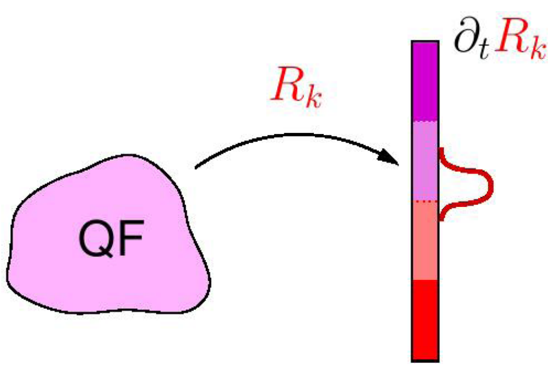
- Cf. microscopic justification: a/d > 1 does not invalidate the microscopic Hamiltonian (as could be suspected from the discussion of weakly interacting gases). The relevant ratio for the validity is $r_{vdW}/d, r_{vdW}/\lambda_{dB} \ll 1$. Feshbach resonances violate the generic relation $r_{vdW}/a \approx 1$: "anomalously large scattering length"

Functional Renormalization Group Approach

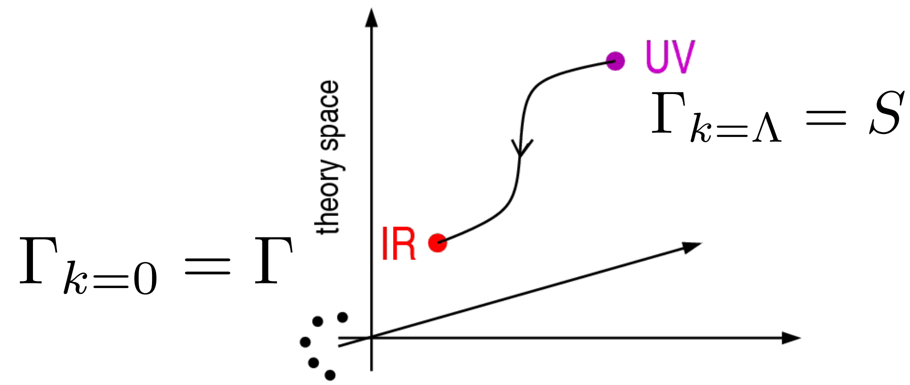
$$\partial_t \Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\,\frac{1}{\Gamma_k^{(2)}[\phi] + R_k}\,\partial_t R_k =$$



Wetterich Equation

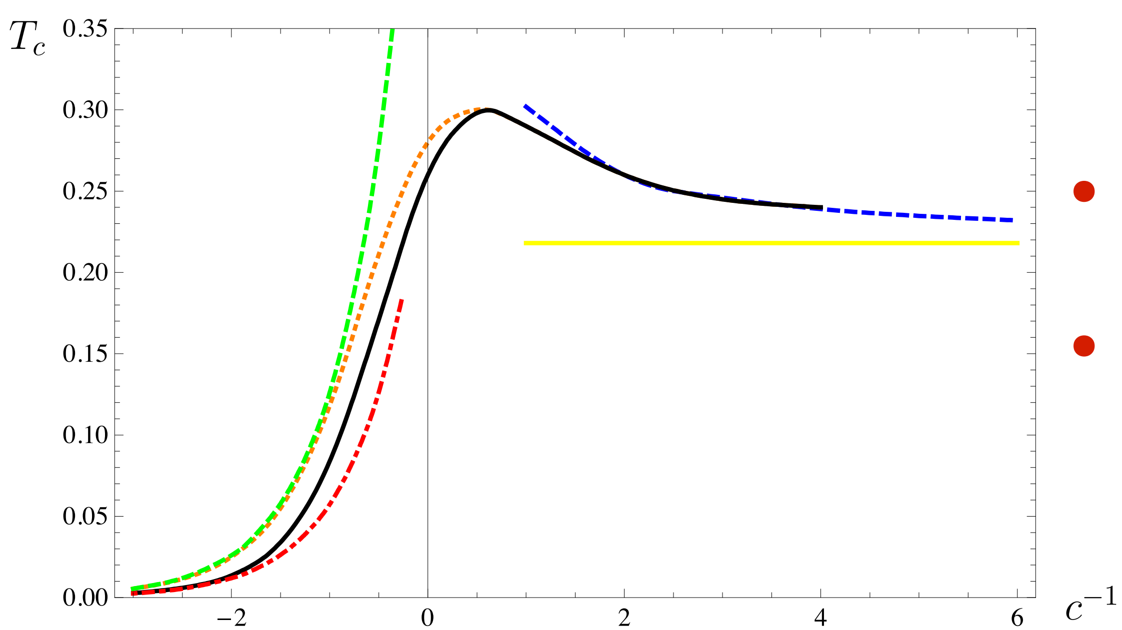


Integrating out quantum and thermal fluctuations



flowing action

Crossover phase diagram



Motivation:

- Transition from well-known mircrophysics to macrophysics
- use FRG to resolve physics at all scales: microphysics, thermodynamics, crticial behavior

Qualitative Picture for BCS-BEC Crossover from FRG

- This first approach is equivalent to an extended mean field theory. It qualitatively describes the finite temperature phase diagram
- But allows for straightforward extensions
- The simplest truncation allows to discuss the building blocks for the evaluation of the problem forming the basis for later refinements

---

- Microscopically, the origin of the BCS-BEC crossover is the expression of a molecular bound state.
- The bosonic bound state formation must thus be contained in any reasonable truncation
- The minimal trunction is a derivative expansion with explicit bosonic degree of freedom

2-component fermion

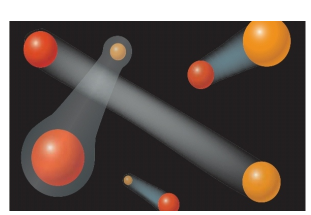
$$\Gamma_k[\psi,\phi] = \int_0^{1/T} d\tau \int d^3x \Big\{ \psi^\dagger (\partial_\tau - \frac{\triangle}{2M} - \mu)\psi - \frac{h_\phi}{2}\Big(\phi^*\psi^T\epsilon\psi - \phi\psi^\dagger\epsilon\psi^*\Big)$$

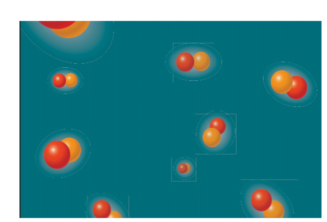
scalar boson field

$$+\phi^*\big(Z_{\phi,k}\partial_\tau - A_{\phi,k}\triangle\big)\phi + U_k(\phi^*\phi) + \ldots\Big\}$$

effective potential for bosons

- Depending on the interaction regime, the boson describes Cooper pairs or tightly bound molecules





The Minimal Approximation Scheme

- The minimal trunction is a derivative expansion with running boson sector

$$\Gamma_k[\psi,\phi] = \int_0^{1/T} d\tau \int d^3x \Big\{ \psi^\dagger \big(\partial_\tau - \frac{\triangle}{2M} - \mu\big)\psi - \frac{h_\phi}{2}\Big(\phi^*\psi^T\epsilon\psi - \phi\psi^\dagger\epsilon\psi^*\Big)$$
$$+\phi^*\big(Z_{\phi,k}\partial_\tau - A_{\phi,k}\triangle\big)\phi + U_k(\phi^*\phi) + \ldots\Big\}$$

- Flow equations:

- The equation for the effective potential (hom. part of eff. action, $U_k(\rho) = T/V\,\Gamma_k(\phi,\phi^* = \mathrm{const.})$

$\rho = \phi^*\phi$ U(1) invariant

$$\partial_t U_k[\rho] = \frac{1}{2}\mathrm{STr}\, \frac{1}{\Gamma_k^{(2)}[\phi] + R_k}\partial_t R_k$$

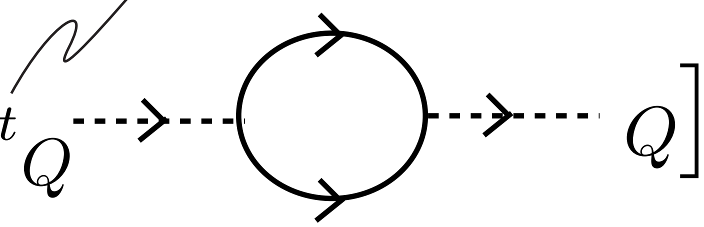
STr -- boson/fermion; internal (spin); external (frequency/momentum)

$$= -\frac{1}{2}\mathrm{Tr}_\psi\, \frac{1}{\Gamma_{\psi,k}^{(2)}[\phi] + R_{\psi,k}}\partial_t R_{\psi,k} + \frac{1}{2}\mathrm{Tr}_\phi\, \frac{1}{\Gamma_{\phi,k}^{(2)}[\phi] + R_{\phi,k}}\partial_t R_{\phi,k}$$

- The equation for the “wave function renormalization”

notation: this derivative acts on explicit (cutoff) k-dependence

$$\partial_t Z_{\phi,k} = \partial_t\Big[\frac{\partial}{\partial(\mathrm{i}\omega)}\Gamma_{\phi,k}^{(2)}\Big]\Big|_{Q=0} = \frac{\partial}{\partial(\mathrm{i}\omega)}\Big[\tilde{\partial}_t$$



$$\Big]\Big|_{Q=0}$$

- And similarly for the “gradient coefficient” $A_{\phi,k}$ $(\partial/\partial(\mathbf{q}^2))$ derivative)

The Flow of the Effective Potential

- We spell out the ingredients of the effective potential explicitly:

$$\partial_t U_k = \underbrace{-\frac{1}{2}\mathrm{Tr}_\psi \, \frac{1}{\Gamma^{(2)}_{\psi,k}[\phi] + R_{\psi,k}} \partial_t R_{\psi,k}}_{\text{fermionic contribution}} + \underbrace{\frac{1}{2}\mathrm{Tr}_\phi \, \frac{1}{\Gamma^{(2)}_{\phi,k}[\phi] + R_{\phi,k}} \partial_t R_{\phi,k}}_{\text{bosonic contribution}}$$

$$\Gamma^{(2)}_{\psi,k} = \Gamma^{(2)}_{\psi} = \begin{pmatrix} -h_\phi \epsilon \phi^* & iq_0 - (\mathbf{q}^{\,2} - \mu) \\ iq_0 + \mathbf{q}^{\,2} - \mu & h_\phi \epsilon \phi \end{pmatrix}$$

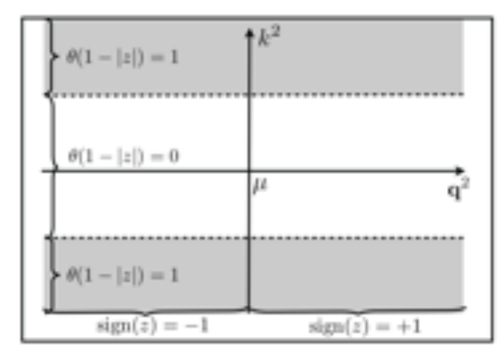
$$\Gamma^{(2)}_{\phi,k} \quad = \quad \begin{pmatrix} U_k' + 2\rho U_k'' + A_{\phi,k}\mathbf{q}^{\,2}/2 & -Z_{\phi,k} q_0 \\ Z_{\phi,k} q_0 & U_k' + A_{\phi,k}\mathbf{q}^2/2 \end{pmatrix}$$

in real (phase-amplitude) basis (ϕ_1, ϕ_2) $\quad (\phi = (\phi_1 + i\phi_2)/\sqrt{2})$

with $U_k' = \partial U_k / \partial \rho$

- NB: Goldstone theorem respected for all k

- Choice of regulator:
- Litim cutoff for bosons and fermions, in the latter case such that the IR limit is on the FS (details see review SD, Floerchinger, Gies, Pawlowski, Wetterich '10)



Building Blocks for the Evaluation

Three key requirements (independent of the implemented approximation scheme)

- We work in grand canonical setting (given chemical potential) but eventually want to consider fixed density.

➡ Construct the equation of state for the density

$$n(\mu) = \ldots$$

- We want to assess the whole phase diagram including the low temperature condensed phase

➡ Implement spontaneous symmetry breaking

- We want to know the results as function of microscopic observables, such as scattering length

➡ Implement proper UV renormalization scheme

The Equation of State

- Thermodynamics:

$$n = -\frac{\partial U}{\partial \mu} = -\frac{\partial U_{k\to 0}}{\partial \mu}$$

approximation: mu dependence of other bosonic couplings neglected

- Flow equation:

$$\partial_k n_k = -\partial_k \frac{\partial U_k}{\partial \mu} \approx \tilde{\partial}_k [\underbrace{\tfrac{1}{2}\mathrm{Tr}_\psi(\Gamma^{(2)}_\psi + R_{\psi,k})^{-1}}_{= n_{\psi,k}} - \underbrace{\tfrac{1}{2}\frac{\partial U'_k}{\mu}\mathrm{Tr}_\phi(\Gamma^{(2)}_{\phi,k} + R_{\phi,k})^{-1}}_{= n_{\phi,k}}]$$

- Interpretation: parts of the trace can be performed (up to conventional normal ordering subtleties)

$$n_{\psi,k} = 2\int \frac{d^3q}{(2\pi)^3}\frac{1}{\exp(E^{(\psi)}_{\mathbf{q},k}/T)+1}$$

Fermi distribution

$$n_{\phi,k} = -\frac{\partial \tilde{U}'_k}{\partial \mu}\int \frac{d^3q}{(2\pi)^3}\frac{1}{\exp(E^{(\phi)}_{\mathbf{q},k}/T)-1}$$

Bose distribution

- with regularized single particle excitation energies

$$E^{(\psi)}_{\mathbf{q},k} = \left[\left(\frac{\mathbf{q}^2}{2M} - \mu + R_{\psi,k}\right)^2 + h_\phi^2\rho\right]^{1/2}$$

cf. weakly interacting fermions

$$E^{(\phi)}_{\mathbf{q},\mathbf{k}} = \left[(\tilde{A}_\phi\mathbf{q}^2 + \tilde{U}'_k + 2\tilde{\rho}\tilde{U}''_k + R_{\phi,k})(\tilde{A}_\phi\mathbf{q}^2 + \tilde{U}'_k + R_{\phi,k})\right]^{1/2}$$

cf. weakly interacting bosons

- we introduced “renormalized” bosonic couplings (interpretation:see later)

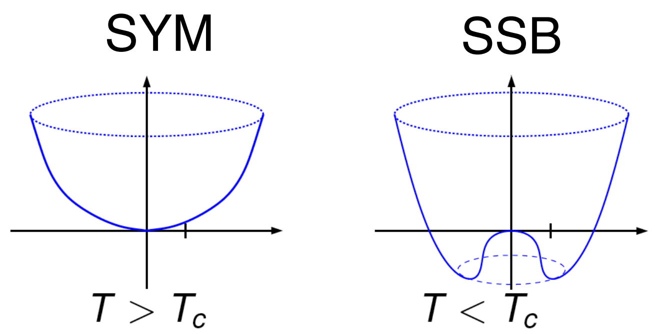
$$\tilde{U}'_k = U'_k/Z_{\phi,k} \qquad \tilde{U}''_k = U''_k/Z^2_{\phi,k} \qquad \tilde{\rho}_{0,k} = Z_{\phi,k}\rho_{0,k}$$

Spontaneous Symmetry Breaking

- Field equation for effective potential (equilibrium condition)

$$\frac{\partial U_k(\rho)}{\partial \phi^*}\Big|_{\mathrm{eq}} \quad = \quad U_k'(\rho)\cdot\phi\big|_{\mathrm{eq}} = 0 \qquad (\rho = \phi^*\phi)$$

- Three types of solution, for the physical limit $k \to 0$



$$\begin{array}{lll} \text{symmetric phase SYM:} & U_k' > 0, & \phi_{0,k} = 0 \\ \text{symmetry broken phase SSB:} & U_k' = 0, & \phi_{0,k} \neq 0 \\ \text{critical point} & U_k' = 0, & \phi_{0,k} = 0 \end{array}$$

- It is sufficient to approximate the rho-dependence of the effective potential further (should be good close to the equilibrium valueρ_0):

$$U_k = m^2_{\varphi,k}(\rho - \rho_{0,k}) + \tfrac{1}{2}\lambda_{\varphi,k}(\rho - \rho_{0,k})^2 + \ldots$$

- with running couplings

SYM	SSB
$m^2_{\phi,k}, \lambda_{\phi,k}$	$\rho_{0,k}, \lambda_{\phi,k}$
$\rho_{0,k} = 0$	$m^2_{\phi,k} = 0$

- NB: SSB criterion works throughout whole crossover. At T=0, SSB occurs for any value of scattering length. Therefore, there is no quantum phase transition, but a crossover phenomenon

The Initial Condition and UV Renormalization

- Problem:
  - Remember: our microscopic formulation is an effective theory valid at low energies and momenta $\Lambda \ll a_{\rm Bohr}^{-1}$
  - But the interaction is formally described by a constant

- Manifestation: there is one strongly running coupling in the UV, the mass term:

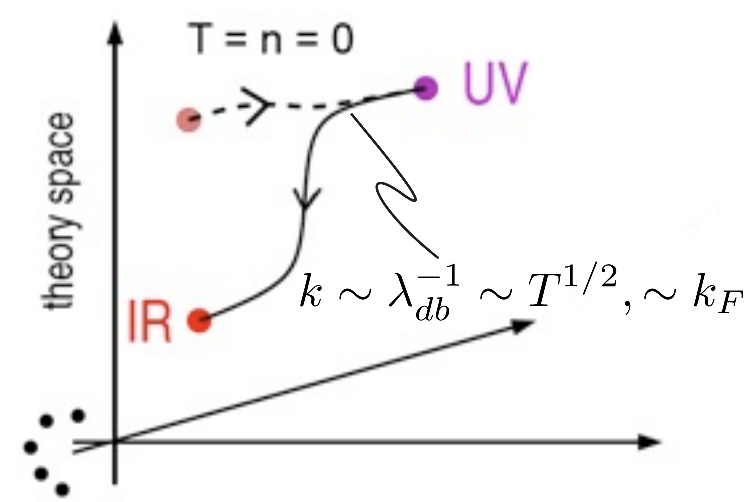
$$m_{\phi,k}^2 \sim k \text{ for } k \to \infty$$

- Ultraviolet Renormalization needed. FRG solution:

  - Experiments probe the "full theory" (with fluctuations), but in the phys. vacuum (two-body scattering)
  - Therefore, project on the physical vacuum via:

$$\Gamma_{k\to 0}(vak) = \lim_{k_F \to 0} \Gamma_{k\to 0}\big|_{T/\varepsilon_F > T_c/\varepsilon_F = \text{const.}} \quad n = \frac{k_F^3}{3\pi^2}$$

- Diluting procedure: $d \sim k_F^{-1} \to \infty$
- Getting cold: $T \sim \varepsilon_F$
- but the dimensionless temperature remains above critical: switch off many-body effects



- Choose UV initial conditions to match IR observables in this limit
- flow for finite n, T deviates from vacuum flow once

$$k \sim \lambda_{db}^{-1} \sim T^{1/2}, \sim k_F$$

The Extended Mean Field Approximation

- Summary: we have a truncation in terms of running bosonic couplings

$$\{m^2_{\phi,k} \text{ or } \rho_{0,k}, \lambda_{\phi,k}, Z_{\phi,k}, A_{\phi,k}\}$$

- and a k-dependent flow equation for the density,

$$n_k = n_{\psi,k} + n_{\phi,k}$$

- Mean field approximation (MFT): for the beta-functions, only take fermion diagrams
- Simplifications:
  - the flow equations for the bosonic couplings can be integrated directly
  - and the equation of state can be solved upon insertion of these solutions

- Discussion

- Bosons are already treated as dynamical, interacting particles in this approximation. We can describe qualitatively the full phase diagram including the transition to the high temperature phase. This is what “extended” refers to.
- within the MFT, no flow for the inverse fermion propagator and the Feshbach coupling is generated (so taking them k-independent is consistent in this framework):

MFT: $$\partial_t \Gamma^{(2)}_{\psi,k} = 0 \qquad \partial_t h_{\phi,k} = 0$$

- Now we discuss the MFT solution at T = 0

The Extended Mean Field Theory of the BCS-BEC Crossover

- The solution for $k \to 0$ produces two self-consistency conditions (omit k = 0 in notation):
  - The UV renormalized gap equation

$$0 = \frac{\partial U}{\partial \rho} = -\frac{1}{a} - \frac{M}{8\pi}\int \frac{d^3q}{(2\pi)^3}\Big[\frac{1}{E_{\mathbf{q}}^{(\mathrm{F})}}\tanh\frac{E_{\mathbf{q}}^{(\mathrm{F})}}{2T} - \frac{1}{E_{\mathbf{q}}^{(\mathrm{F})}}\Big|_{\phi=\mu=0}\Big]$$

using the relation of scattering length and action parameters

$$a^{-1} = -\frac{M}{4\pi}\frac{\nu}{h^2}$$

  - The equation of state

$$n = -\frac{\partial U}{\partial \mu} = n_F + n_B(m_\phi^2, \rho_0, \lambda_\phi, Z_\phi, A_\phi)$$

- Solve for μ and ρ
- Plot as a function of dimensionless crossover parameter

$$\boxed{(a/d)^{-1} = (ak_F)^{-1}}$$

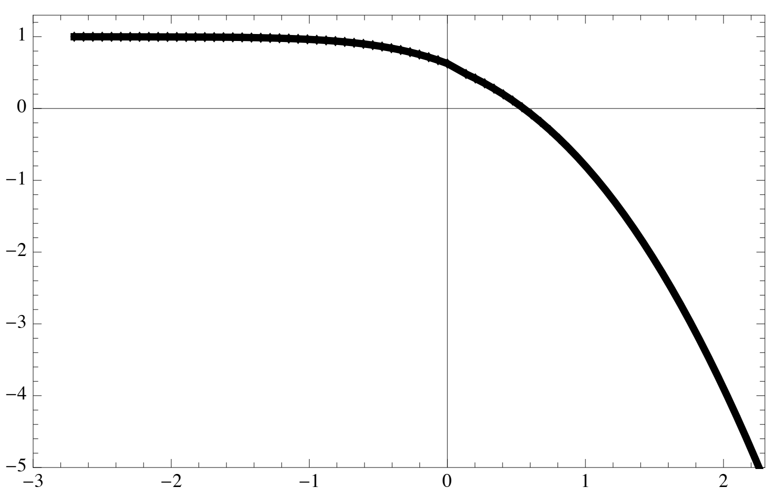
$$n = \frac{k_F^3}{3\pi^2}$$

Fermi momentum

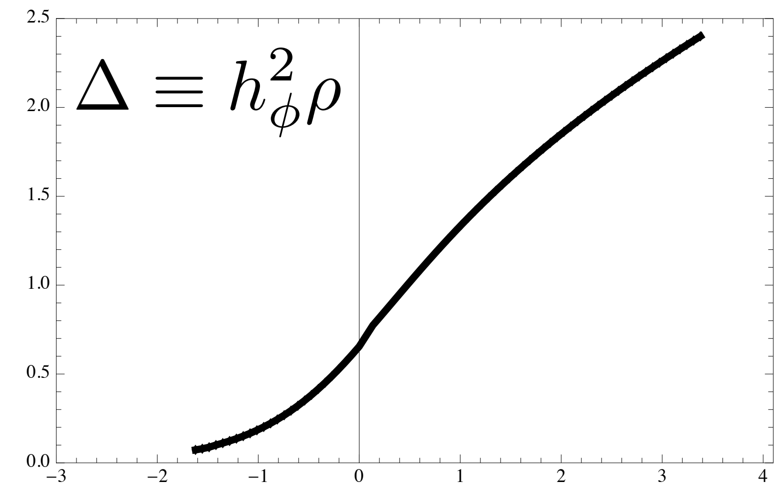
$$\frac{\mu}{\epsilon_F}$$

$$\epsilon_F = \frac{k_F^2}{2M}$$

Fermi energy



0 $(ak_F)^{-1}$



0 $(ak_F)^{-1}$

➡ What do these solutions tell us?

The Extended Mean Field Theory of the BCS-BEC Crossover

- The solution for $k \to 0$ produces two self-consistency conditions (omit k = 0 in notation):

- The UV renormalized ap equation

$$0 = \frac{\partial U}{\partial \rho} = -\frac{1}{a} - \frac{M}{8\pi}\int \frac{d^3q}{(2\pi)^3}\left[\frac{1}{E_{\mathbf{q}}^{(\mathrm{F})}}\tanh\frac{E_{\mathbf{q}}^{(\mathrm{F})}}{2T} - \frac{1}{E_{\mathbf{q}}^{(\mathrm{F})}}\Big|_{\phi=\mu=0}\right]$$

using the relation of scattering lenght and action parameters

$$a^{-1} = -\frac{M}{4\pi}\frac{\nu}{h^2}$$

- The equation of state

$$n = -\frac{\partial U}{\partial \mu} = n_F + n_B(m_\phi^2, \rho_0, \lambda_\phi, Z_\phi, A_\phi)$$

- Solve for μ and ρ

- Plot as a function of dimensionless crossover parameter

$$\boxed{(a/d)^{-1} = (ak_F)^{-1}}$$

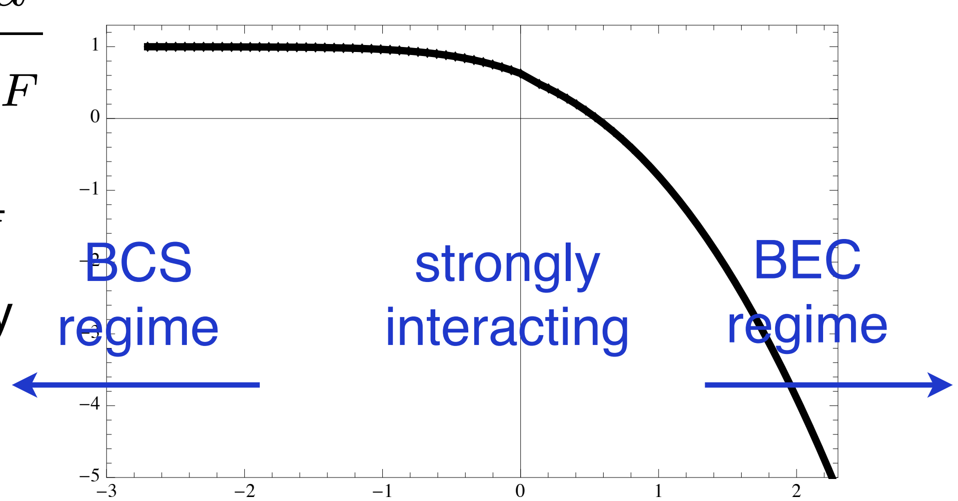
$$n = \frac{k_F^3}{3\pi^2}$$

Fermi momentum

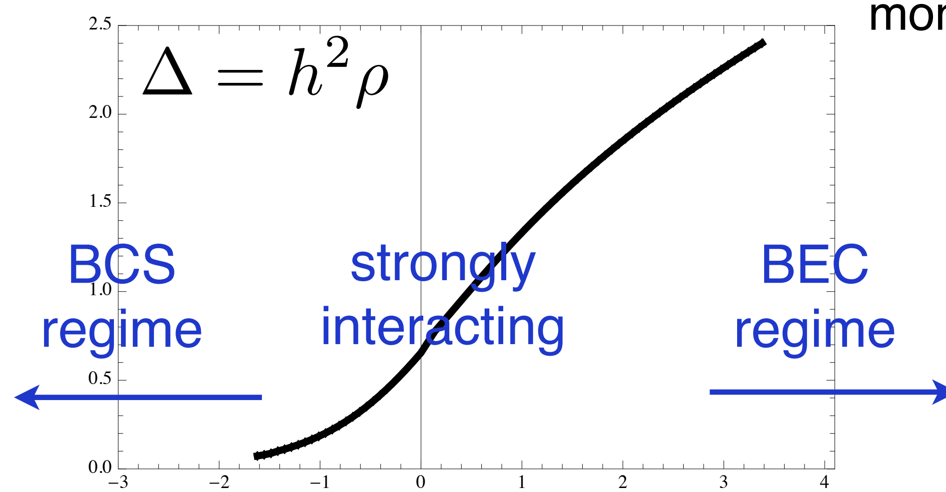
$$\frac{\mu}{\epsilon_F}$$

$$\epsilon_F = \frac{k_F^2}{2M}$$

Fermi energy



0 $\quad (ak_F)^{-1}$



0 $\quad (ak_F)^{-1}$

➡ Discuss the limiting cases!

The Limiting Cases: BCS Limit

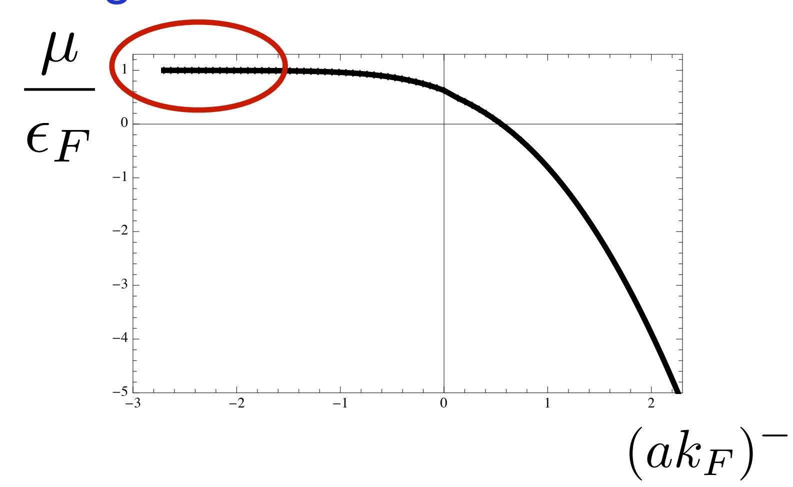
- Solution above: $\frac{\mu}{\epsilon_F} \to 1$
  - ➡ Expression of a Fermi surface, weakly interacting fermion gas is approached

- Simplifications
  - ➡ The EoS reduces to

$$n = n_F + n_B \to n_F$$

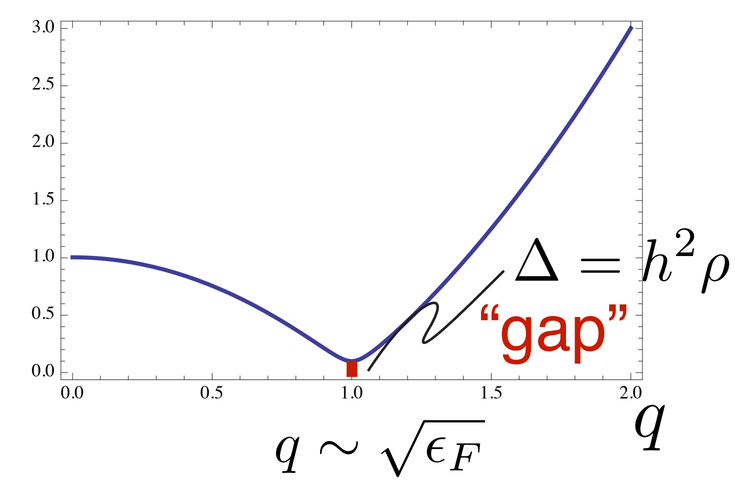
  - ➡ The gap equation can be solved analytically for $\frac{\mu}{\epsilon_F} \to 1$

$$0 = \frac{\partial U}{\partial \rho} = -\frac{1}{a} - \frac{M}{8\pi}\int \frac{d^3q}{(2\pi)^3}\left[\frac{1}{E^{(\mathrm{F})}_{\mathbf{q}}}\tanh\frac{E^{(\mathrm{F})}_{\mathbf{q}}}{2T} - \frac{1}{E^{(\mathrm{F})}_{\mathbf{q}}}\Big|_{\phi=\mu=0}\right]$$



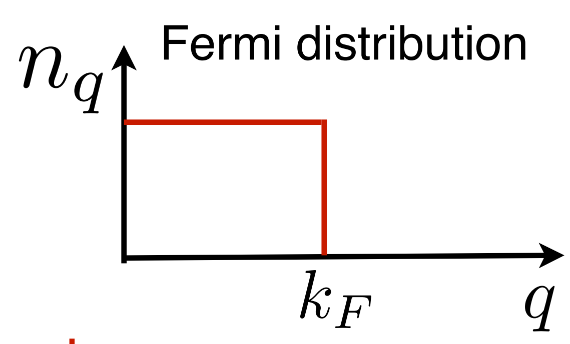
single fermion excitation spectrum

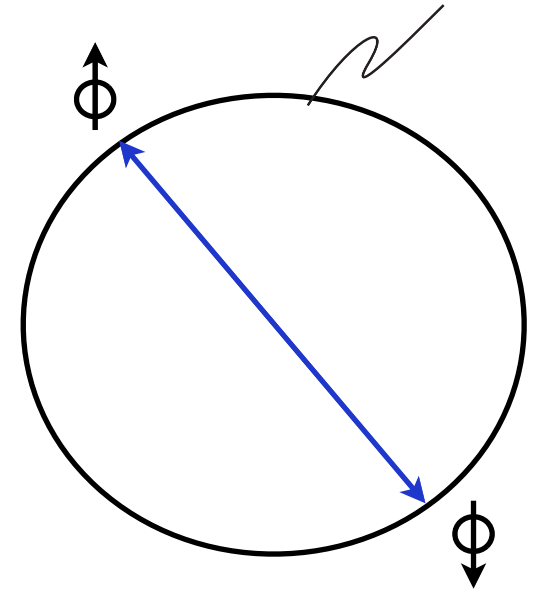
$$E^{(\mathrm{F})}_{\mathbf{q}} = \left[\left(\frac{\mathbf{q}^2}{2M} - \mu\right)^2 + h^2\rho\right]^{1/2}$$



The Limiting Cases: BCS Limit

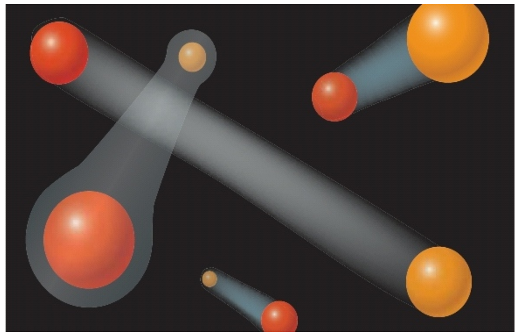
- Result and interpretation:
  - Strongly expressed Fermi surface
    - ➡ Scattering/Pairing highly local in momentum space





Locality in momentum space

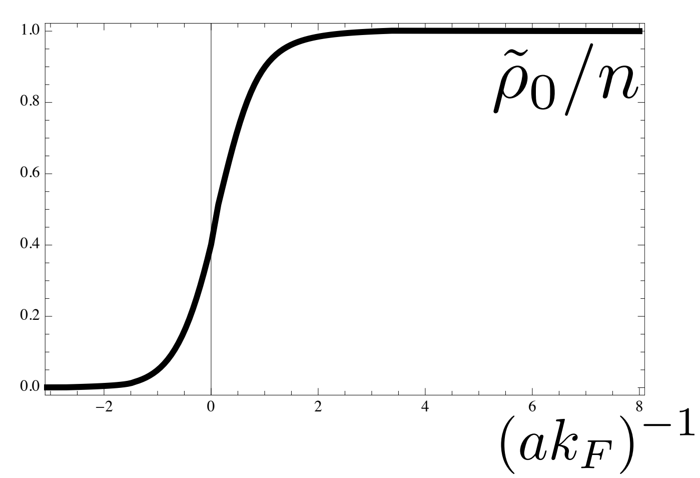
Delocalization in position space



  - The result for the gap:

$$\Delta = 0.61\epsilon_F \, e^{-\frac{\pi}{2ak_F}}$$

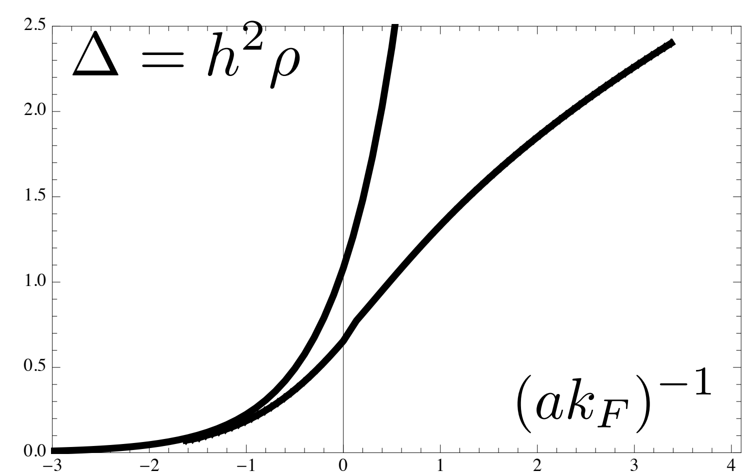
➡ Condensation is very weakly expressed: only Fermions close to Fermi surface contribute



renormalized condensate

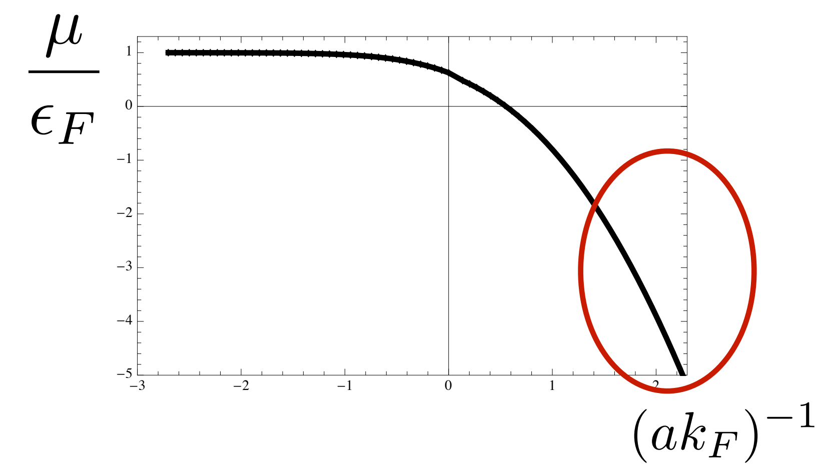
  - Comparison of BCS limit to extended MFT result
    - ➡ Strong deviations from BCS result once

$$(ak_F)^{-1} \sim -1$$



The Limiting Cases: BEC Limit

BEC regime



- Solution above: $\frac{\mu}{\epsilon_F} \to -\infty$

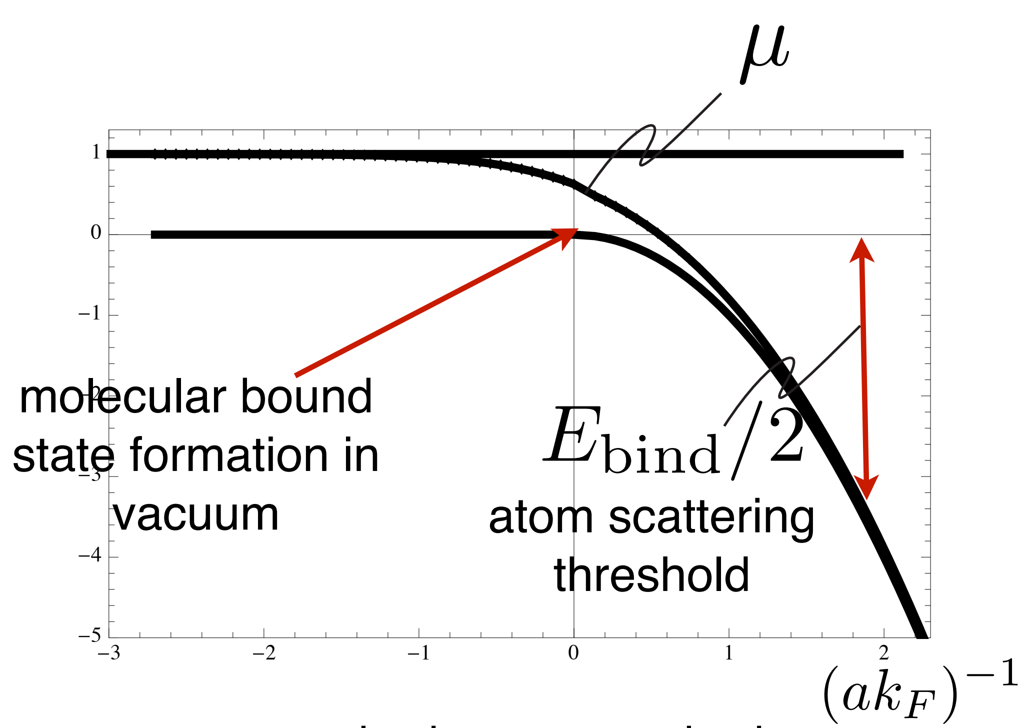
- Simplification of the gap equation: quite drastically,

$$\frac{1}{a} = \sqrt{-\mu \cdot 2M}$$

➡ The density scale k_F (also: temperature) have disappeared from the gap equation

➡ The many-body scales drop out: only two-body physics left!

➡ indeed, comparing to the two-body result obtained as $\Gamma_{k\to 0}(vak) = \lim_{k_F \to 0} \Gamma_{k\to 0}\big|_{T/\epsilon_F > T_c/\epsilon_F = \text{const.}}$



two-body vs. many-body

- Discussion:
  - The chemical potential plays the role of half the binding energy in this limit:

$$\boxed{E_{\text{bind}} = 2\mu = -1/(Ma^2)}$$

  - Smooth crossover terminates in sharp

    "second order phase transition" in vacuum

The Limiting Cases: BEC Limit

- Solution above: $\frac{\mu}{\epsilon_F} \to -\infty$

- Simplification of the fermion density:

  ➡ Strong gap $-\mu$ develops on the normal $(\psi^\dagger \psi)$ sector of the inverse fermion propagator

➡ However, there is a piece from the anomalous part $\psi\psi$ that is independent of $-\mu$

➡ Analysis shows that the fermion density can be written

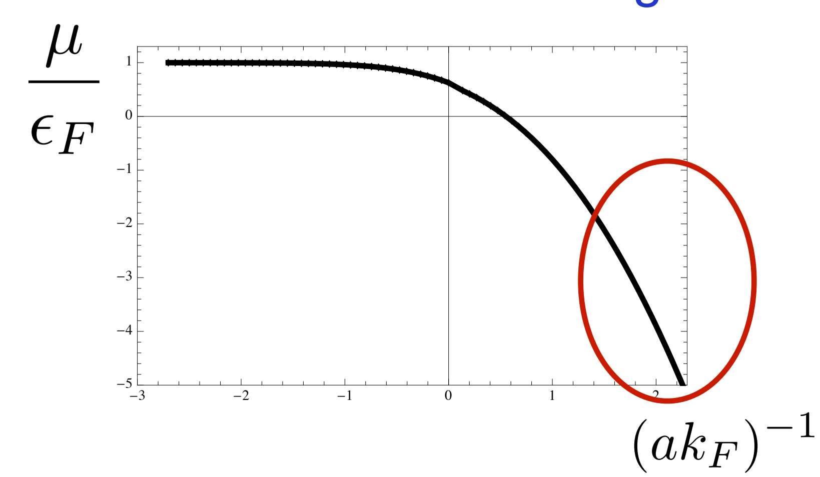
$$\boxed{n_\psi \to 2 Z_\phi \rho_0 \equiv 2\tilde{\rho}_0}$$

wave function renormalization

bare vev for boson action field

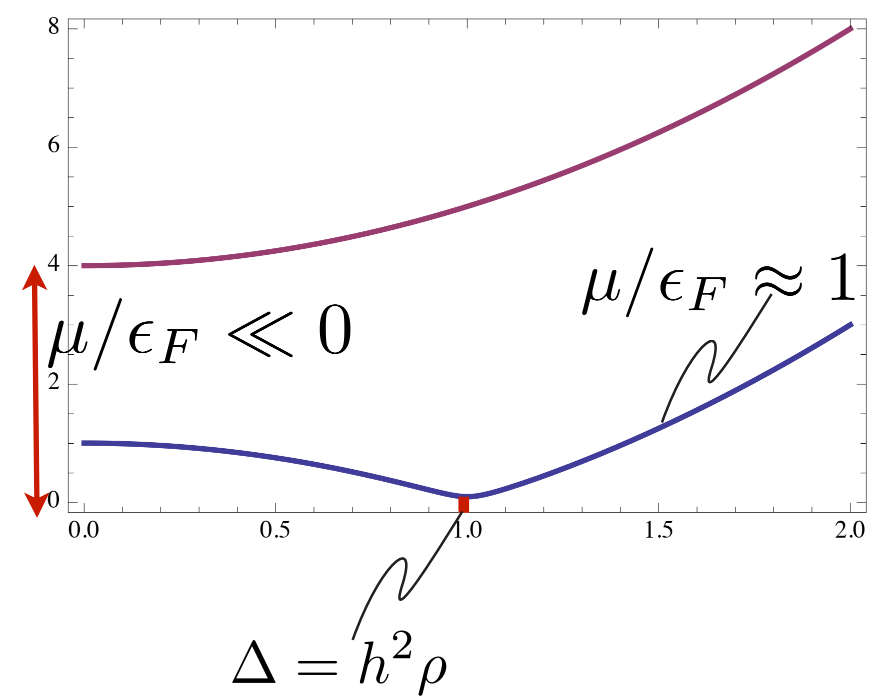
see def. renormalized quantities above!

BEC regime

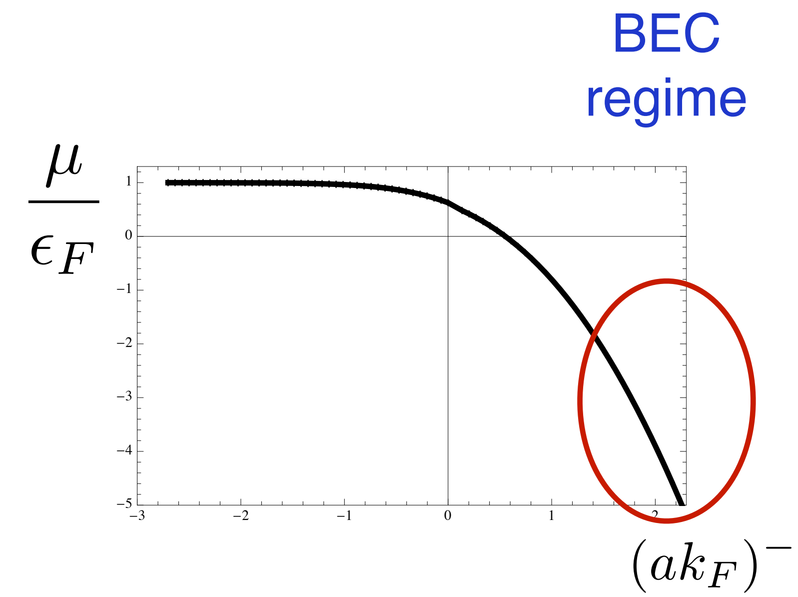


single Fermion excitation spectrum

$$E^{(\mathrm{F})}_{\mathbf{q}} = \left[\left(\frac{\mathbf{q}^2}{2M} - \mu \right)^2 + h^2 \rho \right]^{1/2}$$



The Limiting Cases: BEC Limit



- Solution above: $\frac{\mu}{\epsilon_F} \to -\infty$

- Simplification of the renormalized couplings: Similar to the gap equation, they only feature the scale mu. The renormalized couplings are, for k -> 0,

$$\tilde{m}_\phi^2 = m_\phi^2/Z_\phi \to -2\mu \qquad \tilde{A}_\phi = A_\phi/Z_\phi \to 1/(4M) \qquad \tilde{\lambda}_\phi = \lambda_\phi/Z_\phi^2 \to 2\sqrt{2M\mu} = 2a$$

gap equation

- I.e. for the inverse boson propagator for k -> 0

$$\tilde{\Gamma}_\phi^{(2)} = \Gamma_\phi^{(2)}/Z_\phi \to \begin{pmatrix} 2a\tilde{\rho}_0 & \mathrm{i}\omega + \frac{\mathbf{q}^2}{4M} + 2a\tilde{\rho}_0 \\ -\mathrm{i}\omega + \frac{\mathbf{q}^2}{4M} + 2a\tilde{\rho}_0 & 2a\tilde{\rho}_0 \end{pmatrix}$$

- I.e. for the bosonic contribution to the density

$$n_\phi = -\frac{\partial \tilde{m}'}{\partial \mu}\mathrm{Tr}_\phi \Gamma_\phi^{(2)\,-1}(Q) \to 2\int \frac{d^3q}{(2\pi)^3}\frac{1}{\exp(E_{\mathbf{q}}^{(\phi)}/T) - 1}$$

$$E_{\mathbf{q}}^{(\phi)} = \left[\frac{\mathbf{q}^2}{4M}\left(\frac{\mathbf{q}^2}{4M} + 2a\tilde{\rho}\right)\right]^{1/2}$$

Emergence of an Effective Theory

- Summary: Expressing all quantities in terms of the renormalized quantities gives
  - Renormalized inverse boson propagator

$$\tilde{\Gamma}_{\phi}^{(2)} = \Gamma_{\phi}^{(2)}/Z_{\phi} \to \begin{pmatrix} 2a\tilde{\rho}_0 & \mathrm{i}\omega + \frac{\mathbf{q}^2}{4M} + 2a\tilde{\rho}_0 \\ -\mathrm{i}\omega + \frac{\mathbf{q}^2}{4M} + 2a\tilde{\rho}_0 & 2a\tilde{\rho}_0 \end{pmatrix}$$

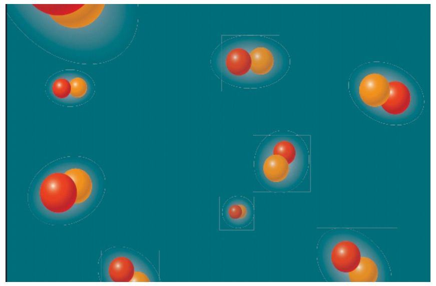
  - Equation of state

$$\boxed{n = 2\tilde{\rho}_0 + 2\int \frac{d^3q}{(2\pi)^3} \frac{1}{\exp(E_{\mathbf{q}}^{(\phi)}/T) - 1}}$$

➡ Reduction to an effective theory of “renormalized” bosonic bound states

- Mass 2M
- Interaction strength 2a
- Atom number 2

Local objects in position space

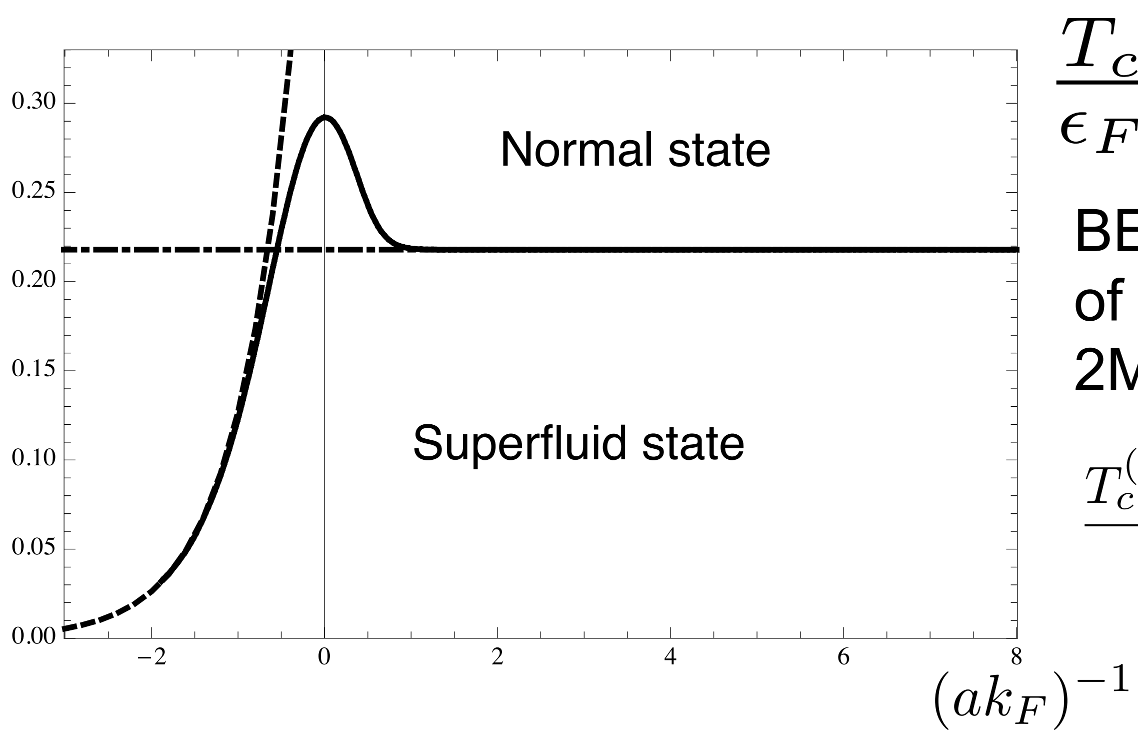


- Discussion:
  - All reference to the concrete value of Z is gone in the renormalized quantities
  - Macroscopic measurements probe the renormalized quantities
  - Microscopic probes can measure Z -> see later!
  - NB: While boson mass and atom number follow from symmetry (Galilei invariance and temporally local gauge symmetry), the interaction strength 2a is an approximation. The exact answer is 0.6a

Finite Temperatures

- So far: Crossover Physics at T=0
- Result for finite temperature phase diagram:

BCS limit:
BCS theory

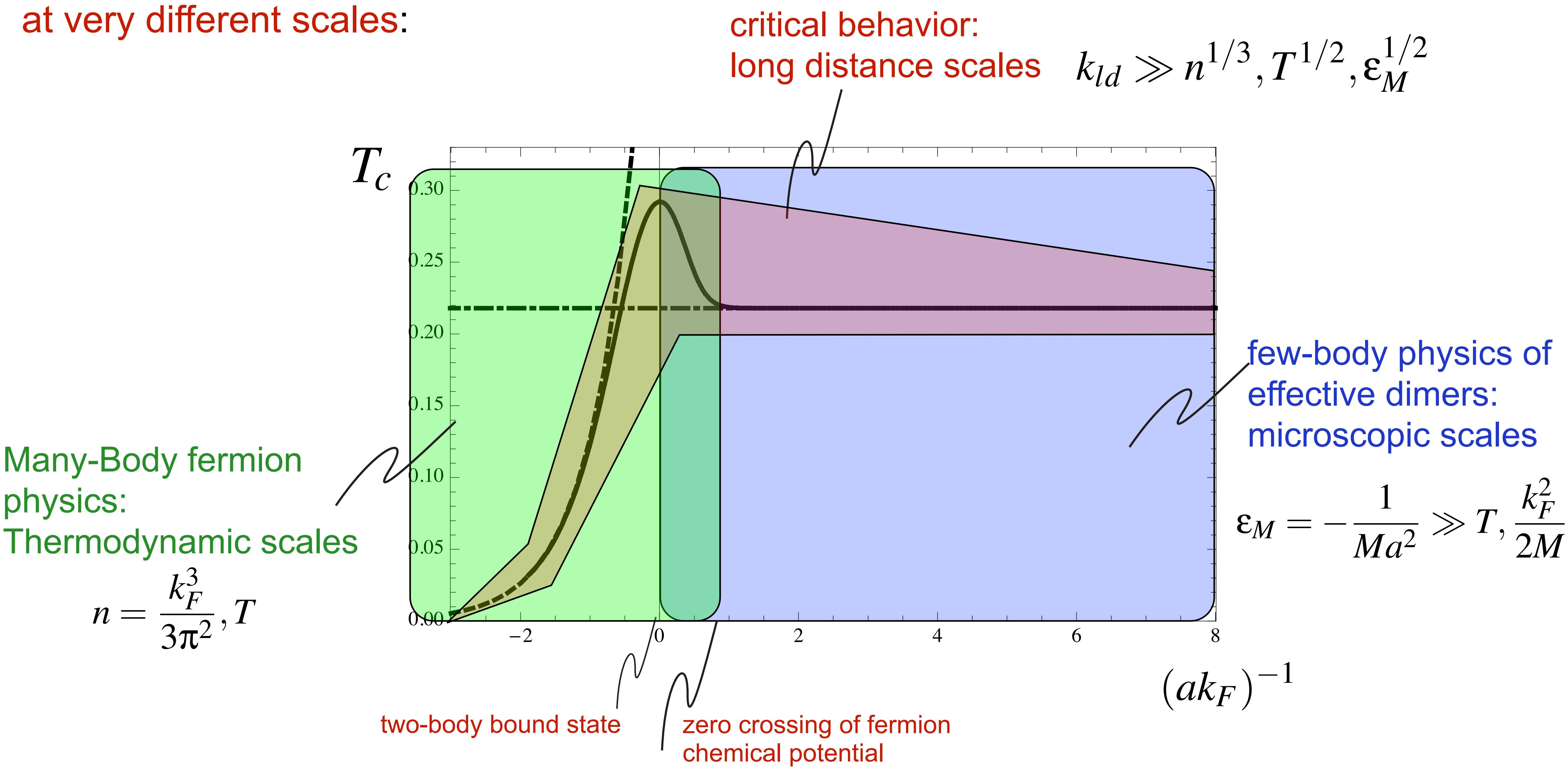


BEC limit: Free bosons of atom number 2, mass 2M

$$\frac{T_c^{(\mathrm{BEC})}}{\epsilon_\mathrm{F}} = 2\pi(6\pi^2\zeta(3/2))^{-2/3} \approx 0.218$$

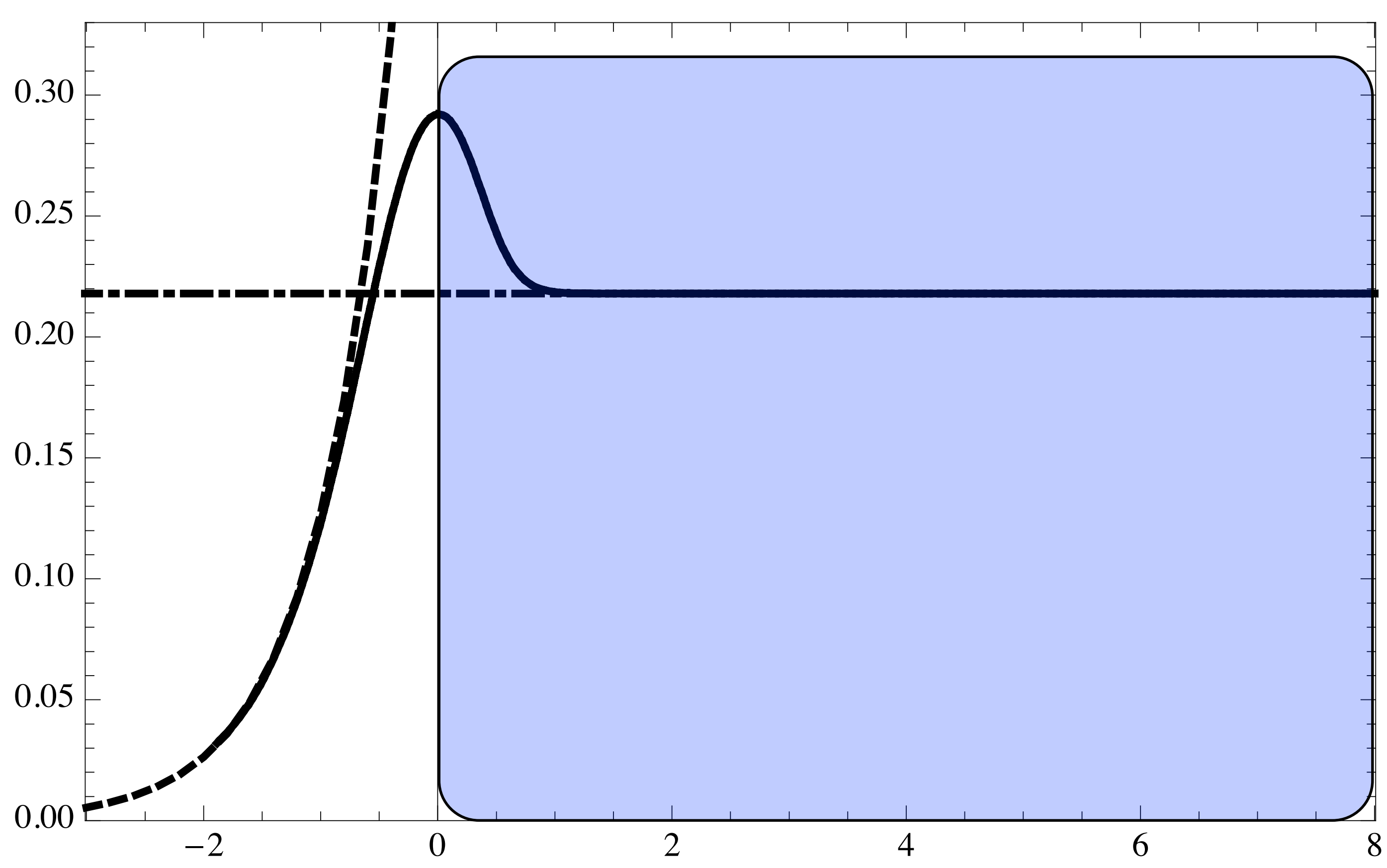
Challenges beyond Mean Field

Beyond mean field effects and challenges
at very different scales:



Strategy: Find an interpolation scheme which incorporates known physical effects in the limiting cases
Methods: t-matrix approaches, 2PI Effective Action, Functional RG, ...

Few-Body Problems from the Functional Renormalization Group



Few-Body Problems from FRG

- Motivation:
  - relevant for the many-body problem (BEC regime)
  - benchmarking of the technique
  - interest in its own right: e.g. Efimov effect in strongly interacting three-body systems (bosons, 3-species fermions), including nonuniversal features out of resonance

---

- Massive diagrammatic simplifications for nonrelativistic few-body problem:
  - Vacuum limit:

$$\Gamma_{k\to 0}(\text{vak}) = \lim_{k_F\to 0} \Gamma_{k\to 0}\Big|_{T/\epsilon_F > T_c/\epsilon_F = \text{const.}}$$

$$n = \frac{k_F^3}{3\pi^2}$$

$$d \sim k_F^{-1} \to \infty$$

$$T \sim \varepsilon_F$$

  - In this constrained limit, remain in symmetric phase: no off-diagonal order

$$\Rightarrow \Gamma_k^{(n,m)} \sim \delta_{n,m}$$

vertex with n in-fields and m out-fields

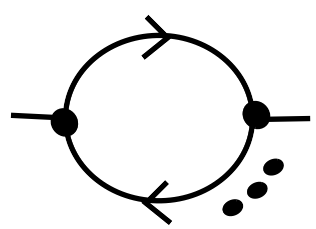
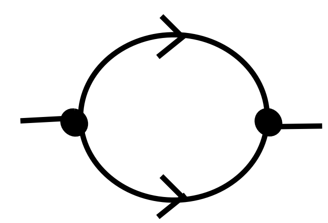
Few-Body Problems from FRG

- In particular, inverse propagators diagonal; the “masses” are semi-positive (stable gs)

$$n + m = 2 : \qquad \Gamma_k^{(2,0)} = \Gamma_k^{(0,2)} = 0 \qquad \Gamma_k^{(1,1)}(Q=0) \geq 0$$

  - physical interpretation: no nonrelativistic antiparticles
  - NB: e.g. Fermi surface $\mu > 0$ thus $\epsilon_{\mathbf{q}} - \mu$ has no definite sign (-> particle-hole fluctuations)

- Poles in a definite half-plane of the complex plane. Thus, diagrams with cyclic flow direction vanish (residue theorem)

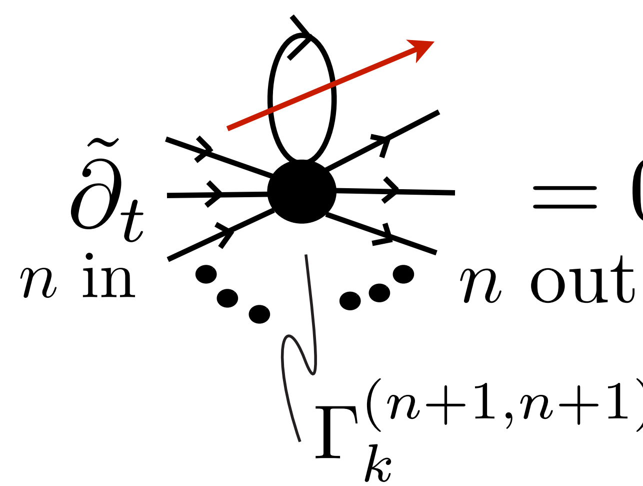
e.g. $\tilde{\partial}_t$  $\to 0$ but $\tilde{\partial}_t$  $\neq 0$

- Implication: nonrelativistic n-body problem solvable within vertex expansion to order n

  - Flow of the diagonal vertices in vertex expansion:

one-loop structure: the highest vertex is:

$$\partial_t \Gamma_k^{(n,n)} = \sum_m^{n+1} \tilde{\partial}_t \Gamma_k^{(m,m)}$$

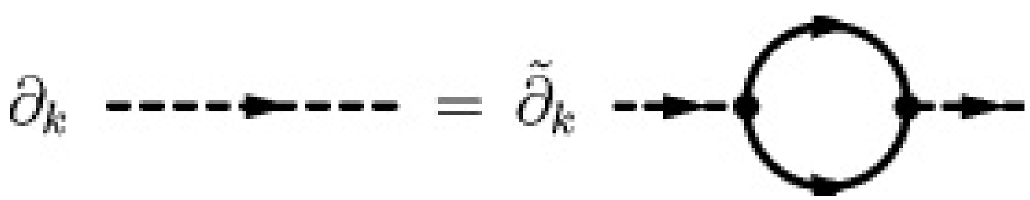
$\tilde{\partial}_t$ 

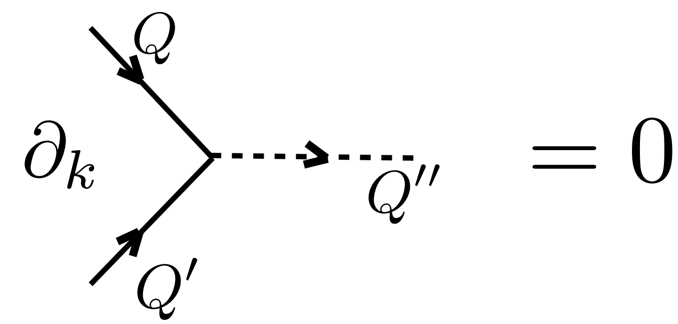
$= 0$

Vertex Expansions in Vacuum

- Vertex expansions keeping full momentum dependence are manageable for specific scattering problems if kinematic simplifications can be used (SD, Krahl, Scherer ’08; Floerchinger, Schmidt, Moroz, Wetterich ’09)
- The resulting exact solutions can be compared to simplified truncations to get analytical insights (Moroz, Floerchinger, Schmidt, Wetterich ’09 three-body (Efimov) problem; Birse et al. four-body problem ’10; see review by Floerchinger, Moroz, Schmidt arxiv.1102.0896)

- The solution of the fully momentum dependent two-body problem (NB: fermionic sector does not renormalize):

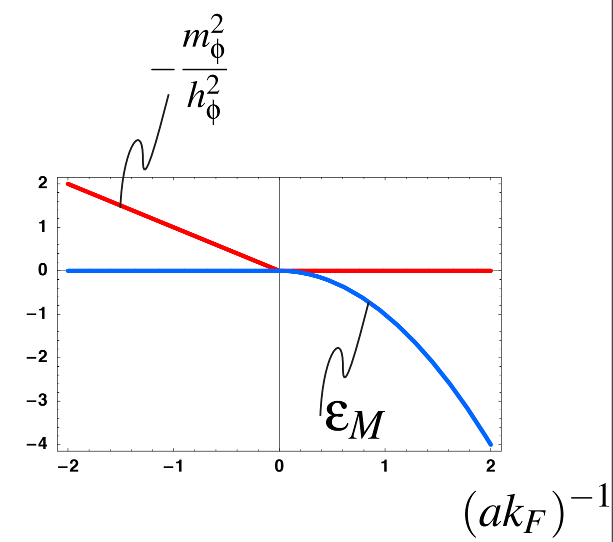




- Solution for 1/a > 0: dimer propagator (UV renormalization: see above)

$$\Gamma^{(2)}_{k=0}(Q) = \frac{h_\varphi^2}{8\pi}\left(-a^{-1} + \sqrt{\mathrm{i}\omega + q^2/4M - \epsilon_M/2}\right)$$

- Binding energy: $\epsilon_M = -1/(Ma^2)$ $\qquad \Gamma^{(2)}_{k=0}(Q=0) \stackrel{!}{=} 0$ gapless dimer propagation



- Relation to derivative expansion: large binding energy

$$\Gamma^{(2)}_{k=0}(Q) \approx Z_\varphi\left(\mathrm{i}\hat{\omega} + \frac{\hat{q}^2}{4M}\right) \qquad Z_\varphi = \frac{h_\varphi^2 a}{32\pi}$$

-> (euclidean) bosons with mass 2M

bound state formation: blue line signals ground state. Negative 1/a: two unbound atoms; positive 1/a: dimer bound state

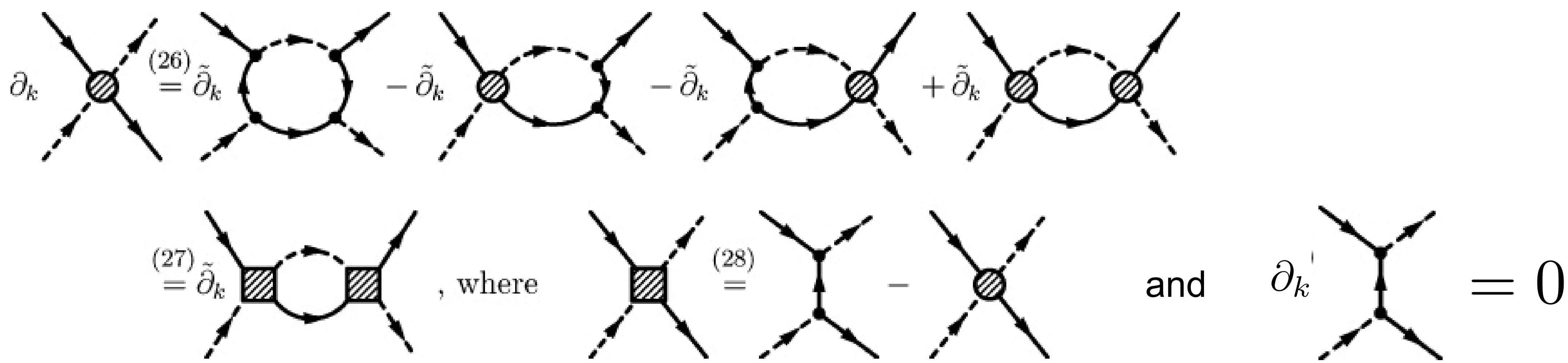
Three-Body Problem

- The solution of momentum dependent three-body (atom-dimer) problem from FRG (SD, Krahl, Scherer '08) (equivalent to solution of STM equation in nuclear physics (Skorniakov, Ter-Matriosian '56))

  - add a fully momentum atom-dimer vertex to the truncation:

$$\Delta\Gamma_k = \int_{Q_1,\ldots Q_4} \delta(Q_1 + Q_2 - Q_3 - Q_4)\delta\lambda_3(Q_1, Q_2, Q_3)\varphi(Q_1) \times \psi(Q_2)\varphi^*(Q_3)\psi^\dagger(Q_4). \tag{8}$$

  - Flow equation (fermion-boson-flow):



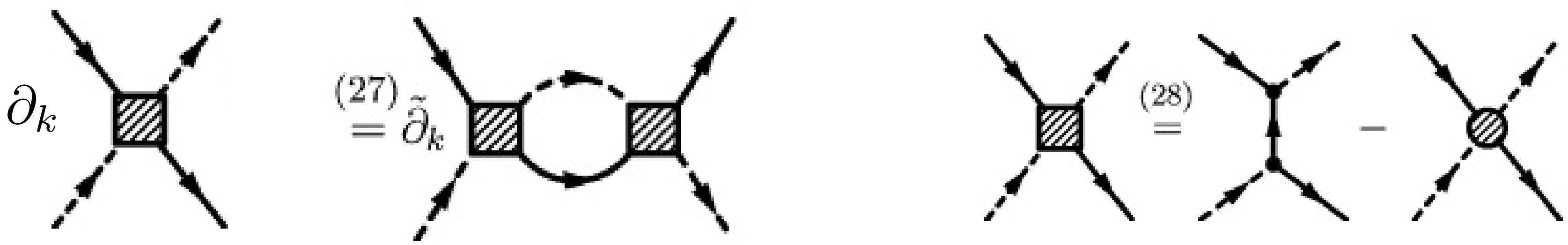
not 1PI

-> can be brought to quadratic form!

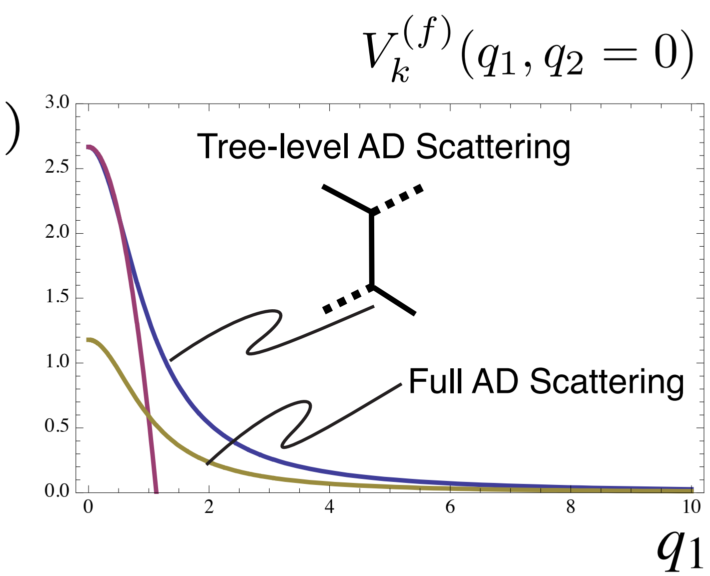
- Using kinematic simplifications and projection to zero angular momentum partial waves (s-wave projection, angular averaging): Equation can be reduced to a matrix differential equation

Three-Body Problem

- Matrix flow equation for s-wave scattering vertex (fermion-boson flow)



$$\partial_k V_k^{(f)} = +V_k^{(f)} M_k V_k^{(f)} \qquad V_k^{(f)} = V_{0,k}^{(f)} - \delta V_k^{(f)}$$



- solution for $k \to 0$

$$V_0^{(f)} = (1 - V_{0,0}^{(f)} M_k)^{-1} V_{0,0}^{(f)} \qquad \text{STM Equation}$$

- Observable of interest: atom-dimer scattering length $a_{ad} = V_k^{(f)}(q_1 = q_2 = 0)$

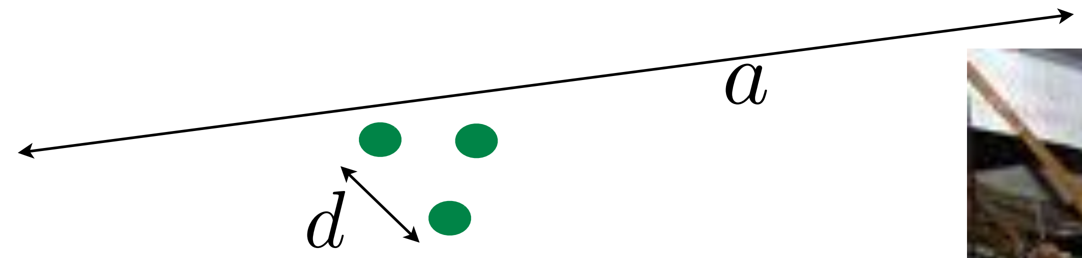
	full solution	pointlike truncation	tree level
$\frac{a_{ad}}{a}$	1.12	1.72	8/3 = 2.67

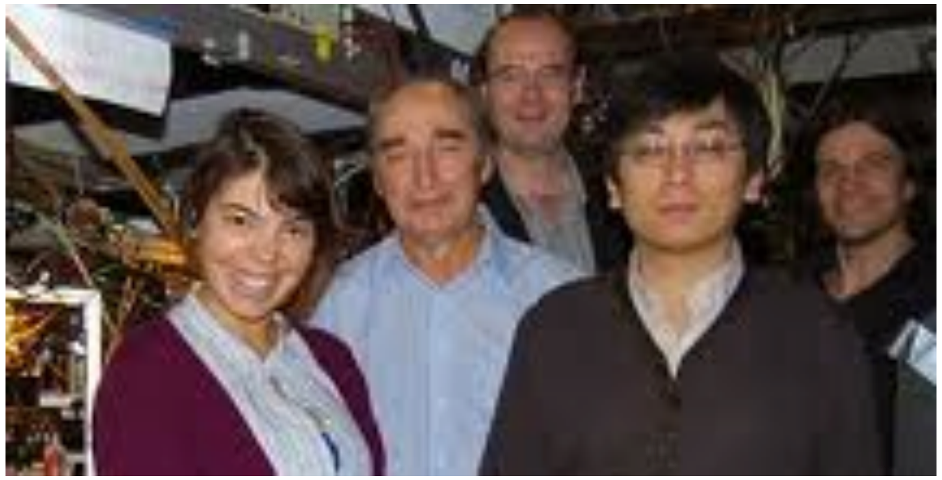
- Replacing fermions by bosons, obtain (Moroz, Floerchinger, Schmidt, Wetterich ’09)

$$\partial_k V_k^{(b)} = -V_k^{(b)} M_k V_k^{(b)} \qquad V_k^{(b)} = 2(V_{0,k}^{(f)} + \delta V_k^{(f)})$$

- These signs have important physical consequences

Efimov Effect





Efimov with Innsbruck experimentalists, confirming his theory (Kraemer ’06, Knoop ’09)

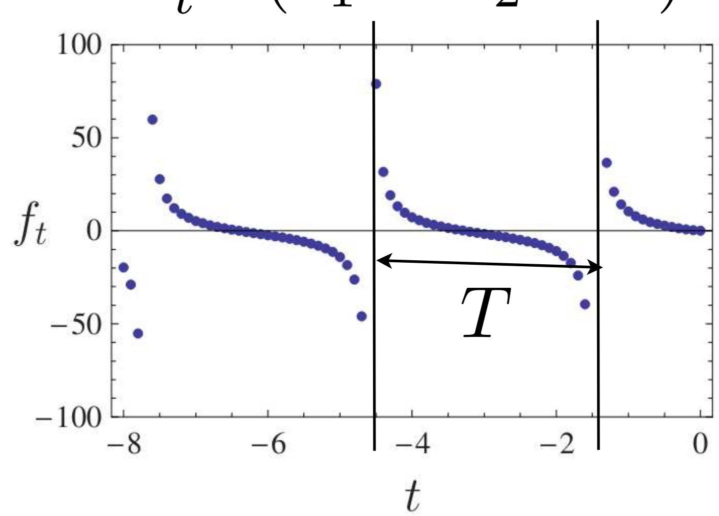
- Vitaly Efimov ’70,’73:
  - Schrodinger Equation of three resonantly interacting identical bosons maps to scattering in 1/r^2 potential at short distances
  - This potential has discrete exp-spaced spectrum with

$$\boxed{\frac{E_{n+1}}{E_n} = \exp(-2\pi/s_0)} \qquad s_0 \approx 1.00624$$
$$n = 1, 2, \dots$$

- Qualitative and quantitative behavior can be found from the FRG approach along the lines above

- Result for atom-dimer scattering amplitude for identical bosons:

$$V_t^{(b)}(k_1 = k_2 = 0)$$



log-spaced Efimov resonances in the RG flow: divergence of the scattering amplitude signals a new trimer bound state

- In RG Language, the Efimov effect is understood as an RG limit cycle (as opposed to an RG fixed point) with “period”

$$T = \pi/s_0$$

Efimov Effect

- Insight can be gained from the pointlike limit (S. Moroz et al. '09) (approximate matrix by single entry $\lambda_{3,k} = \delta V_k^{(s)}(k_1 = k_2 = 0)$)
  - Flow of dimensionless scattering amplitude $\tilde{\lambda}_{3,t} = \lambda_{3,t} k^2$

$$\partial_t \tilde{\lambda}_{3,t} = \alpha \tilde{\lambda}_{3,t}^2 + \beta \tilde{\lambda}_{3,t} + \gamma$$

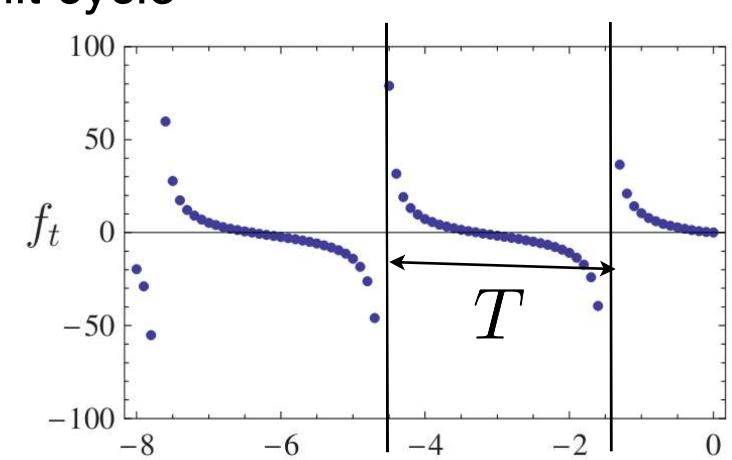
$$\alpha = -c_p/4 \qquad \beta = -c_p + 2 \qquad \gamma = -c_p \qquad c_p = 4(3+p)/(\sqrt{3}\pi)$$

  - Solution for infrared flow $t \to -\infty$

$$\tilde{\lambda}_{3,t} \sim \tanh(\sqrt{D}t/2)$$ with discriminant $$D = \sqrt{\beta^2 - 4\alpha\gamma}$$

  - fermions $p = -1: \quad D > 0 \quad \Longrightarrow$ convergence to IR fixed point
  - bosons $p = 1: \quad D < 0 \quad \Longrightarrow$ convergence to IR limit cycle

$$\tanh(\mathrm{i}\sqrt{-D}t/2) = \tan(\sqrt{-D}t/2)$$

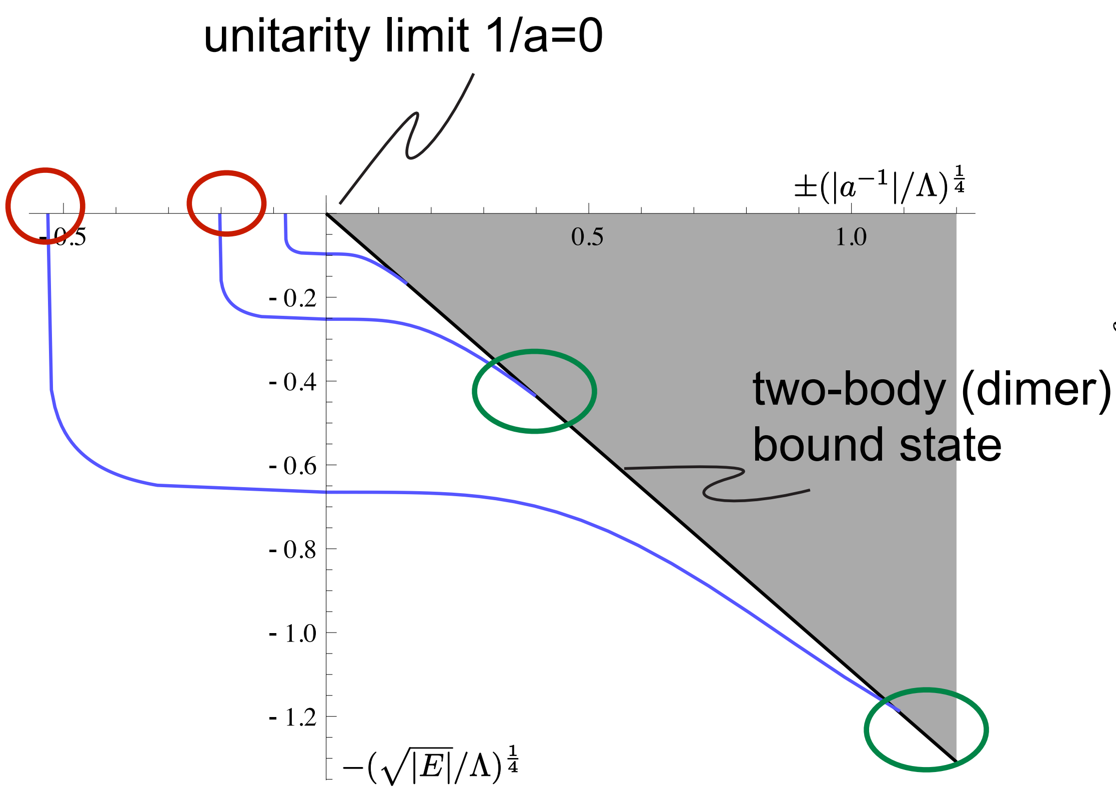


- Quantitatively

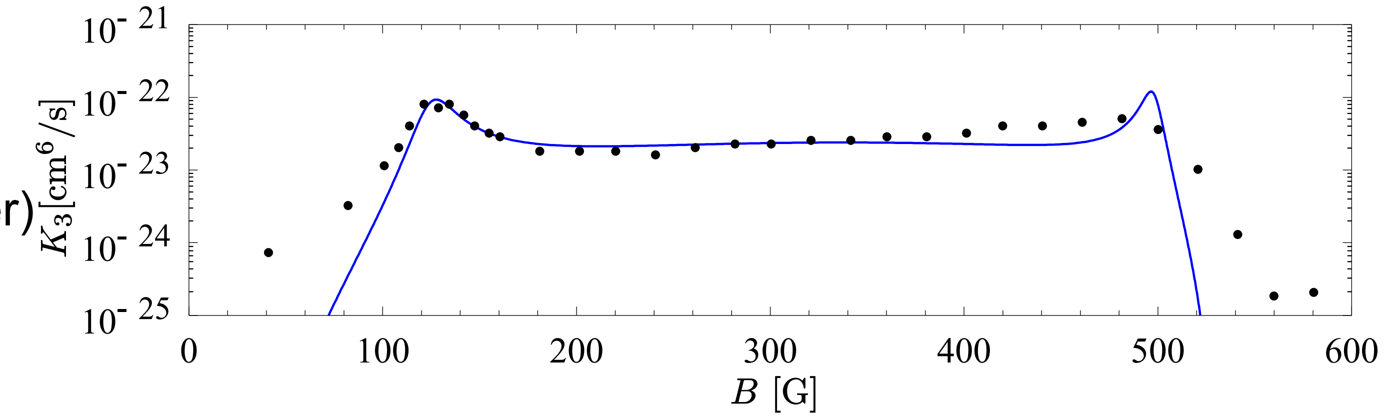
$$s_0 \approx 1.393$$ exact: $$s_0 \approx 1.00624$$

Connection to Experiments

- Three-component fermions (three hyperfine states) can exhibit Efimov effect as well (no Pauli blocking) (Braaten & '09, Naidon,Ueda '09, Floerchinger & '09)



3-body loss rate vs. magnetic field (blue: FRG result (Floerchinger & '09); dots: experimental data (Ottenstein & '08))



Efimov spectrum from FRG
(from review arxiv.1102.0896):

- lowest Efimov bound state determined by short-distance physics
- universal bound state sequence at unitarity
- Efimov resonances at three-atom continuum (red circles and atom-dimer threshold (green circles)
- resolve full Efimov tower, also away from resonance

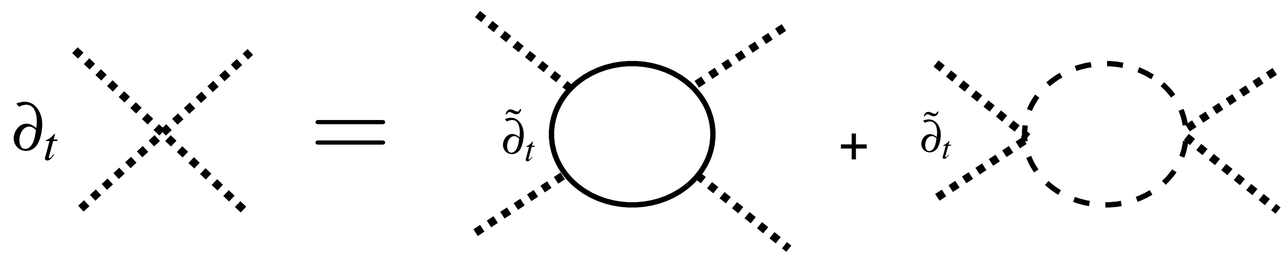
Comparison to experiment:

- Efimov state forms at the three-atom threshold
- There the system shows enhanced loss features (new 3-body decay channels open up)

See Talk by S. Moroz!

The four-body problem and connection to thermodynamics

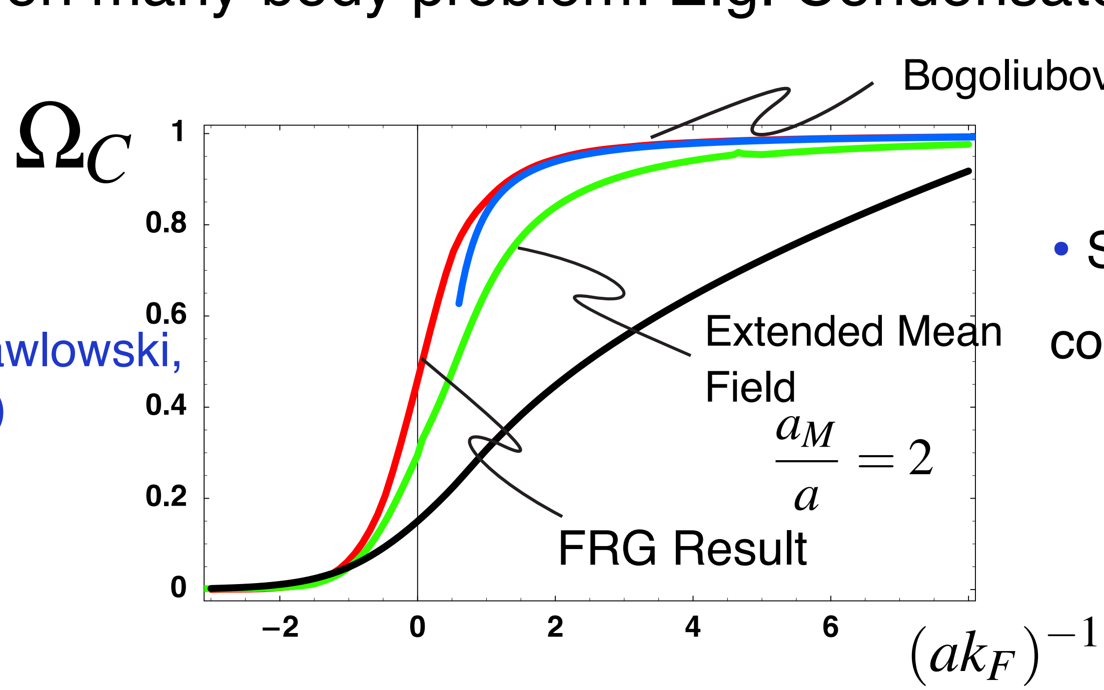
- Vacuum: dimer-dimer scattering on BEC side a > 0



Exact solution from 4-body Schrodinger equation (Petrov, Salomon, Shlyapnikov ’04):

dimer-dimer scattering length $\frac{a_M}{a} =$ 2 0.75 0.6

- Impact on many-body problem: E.g. Condensate Fraction at T=0:

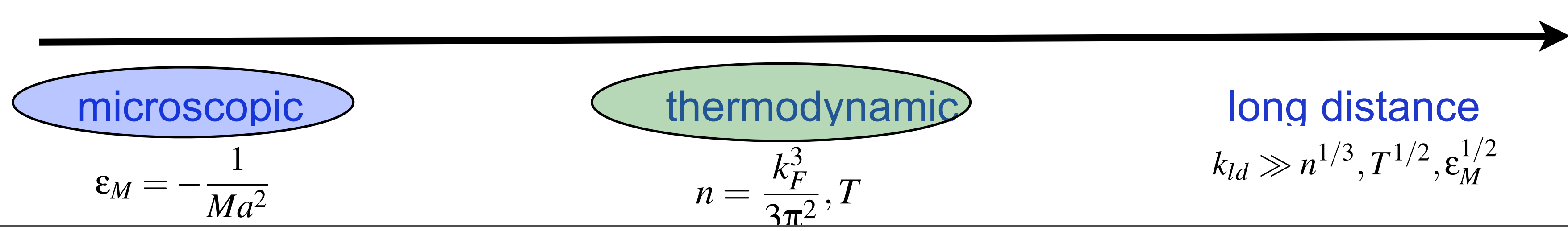


(SD, Gies, Pawlowski, Wetterich ’07)

- Systematic improvement possible: all possible local couplings for dimer-dimer vertex (Birse, Walet, Krippa ’10):

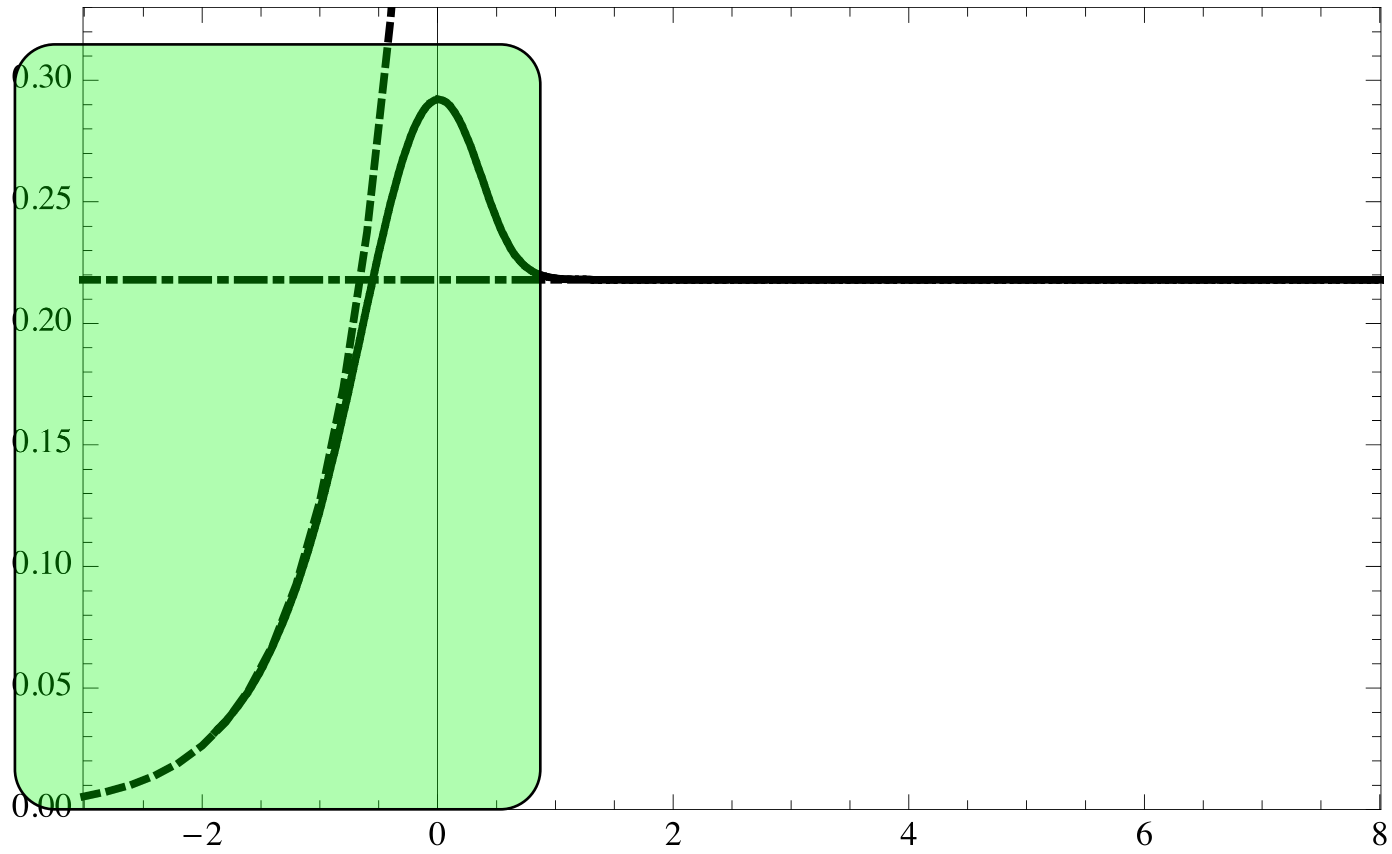
$$\boxed{a_M/a = 0.58 \pm 0.02}$$

- Picture: Tightly bound, weakly interacting molecules deep on BEC side: effective pointlike dof.s interacting via effective molecular scattering length



Beyond Mean Field Many-Body Effects in the BCS-BEC Crossover

Floerchinger, Scherer, SD, Wetterich ’08; Scherer, Floerchinger, Wetterich ‘10

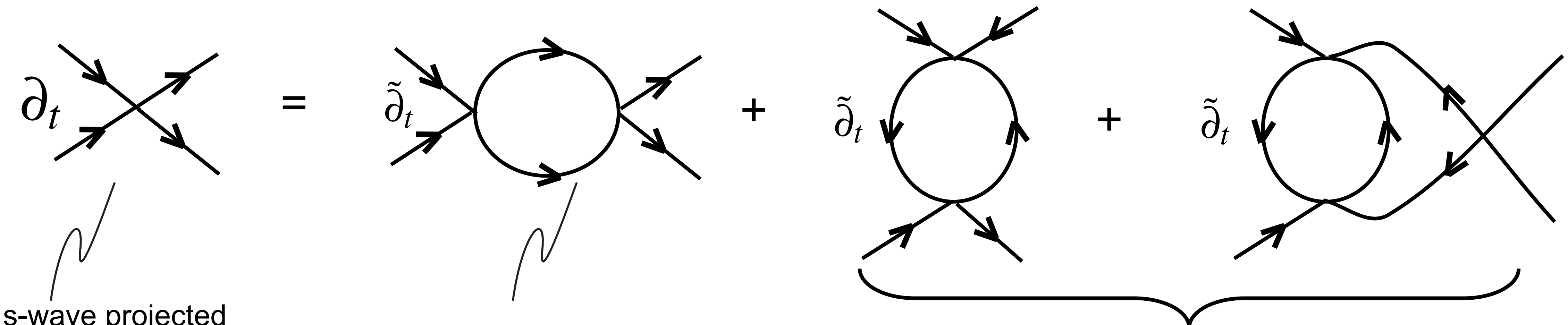


Many-Body Fermion Physics

Particle-Hole Fluctuations for weakly interacting fermions:

- Purely fermionic description $$S[\psi,\phi] = \int d\tau \int d^3x \left\{ \psi^\dagger \left(\partial_\tau - \frac{\triangle}{2M} - \mu \right) \psi + \frac{\lambda}{2} (\psi^\dagger \psi)^2 \right\}$$
- Simple RG Equation beyond log-divergent contribution:

particle-particle channel | particle-hole channels



s-wave projected
4-fermion vertex

$\sim \log T \quad \text{for} \quad T \to 0$

regular as $T \to 0$

- Screening effect with impact on critical temperature at weak interaction
  - Thouless criterion $\lambda^{-1}_{k\to 0}(T,n) = 0$
  - result

$$\frac{T_c^{(BCS)}}{\epsilon_F} = 0.61 e^{-\frac{\pi}{2|a|k_F}}, \qquad \frac{T_c^{(BCS)}}{T_c^{(Gorkov)}} = 2.2 \qquad \text{Gorkov effect}$$

microscopic	thermodynamic	long distance
$\epsilon_M = -\frac{1}{Ma^2}$	$n = \frac{k_F^3}{3\pi^2}, T$	$k_{ld} \gg n^{1/3}, T^{1/2}, \epsilon_M^{1/2}$

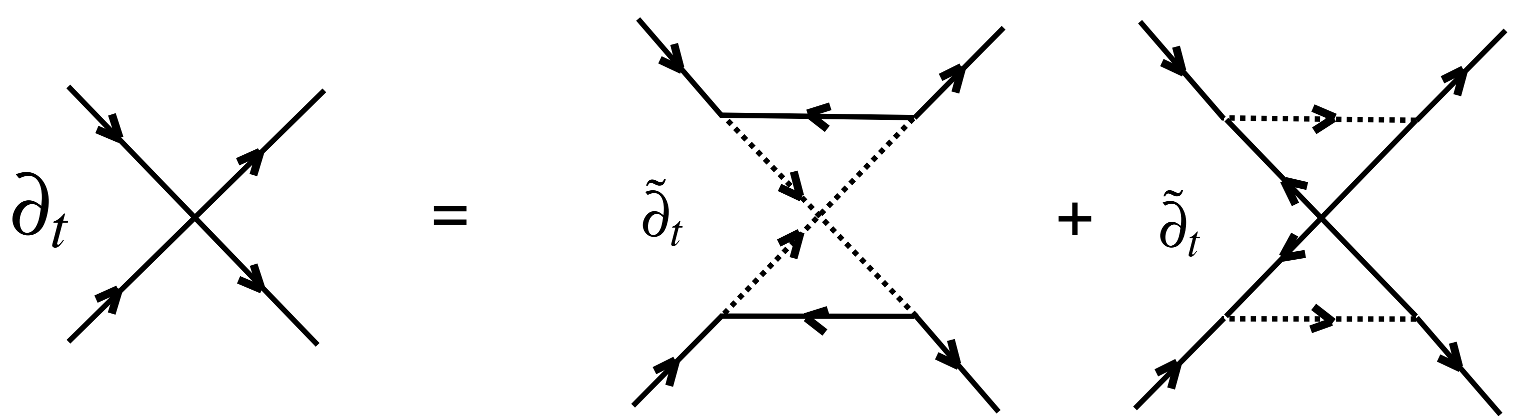
Many-Body Fermion Physics

- Hubbard-Stratonovich transformation: Decoupling into particle-particle channel
- essential: describe the bound state generation
- how to reconstruct the lost particle-hole channel?
- Study flow of newly generated 4-fermion vertex

- extend truncation: $$\Delta\Gamma_k = \int \lambda_{\psi_k} (\psi^\dagger \psi)^2$$

- initial condition: $$\lambda_{\psi_{k=\Lambda}} = 0$$

- flow:



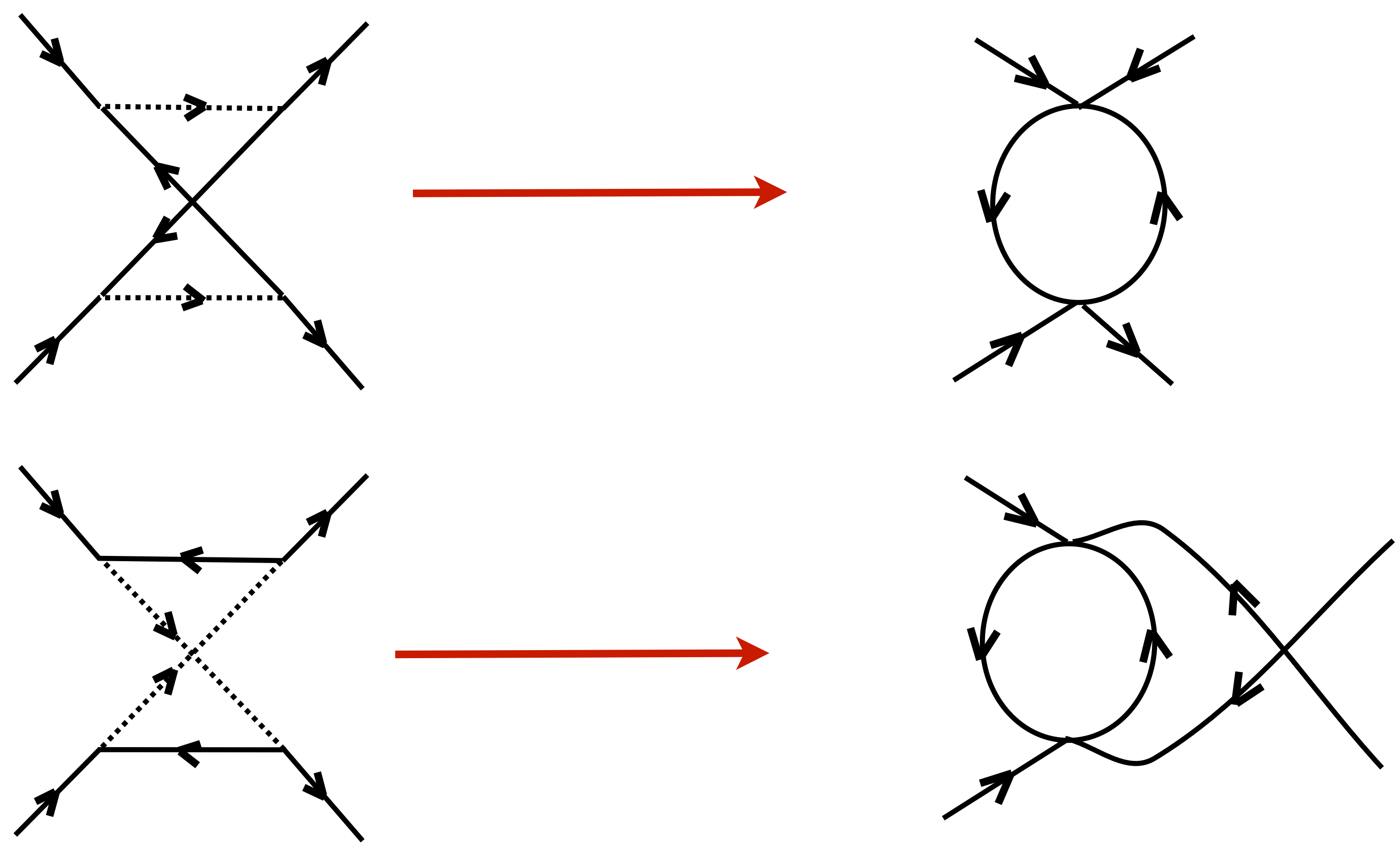
✓ s-wave projected
✓ included via
rebosonization technique
(Gies, Wetterich '02)

microscopic	thermodynamic	long distance
$\epsilon_M = -\frac{1}{Ma^2}$	$n = \frac{k_F^3}{3\pi^2}, T$	$k_{ld} \gg n^{1/3}, T^{1/2}, \epsilon_M^{1/2}$

Many-Body Fermion Physics

Interpretation

- assume massive bosons $\quad P_{\phi,k}(Q) \approx m^2_{\phi,k}$
- contract boson lines $\quad \lambda_{ph,k} \approx \dfrac{h^2_{\phi,k}}{m^2_{\phi,k}}$



microscopic

$$\epsilon_M = -\frac{1}{Ma^2}$$

thermodynamic

$$n = \frac{k_F^3}{3\pi^2}, T$$

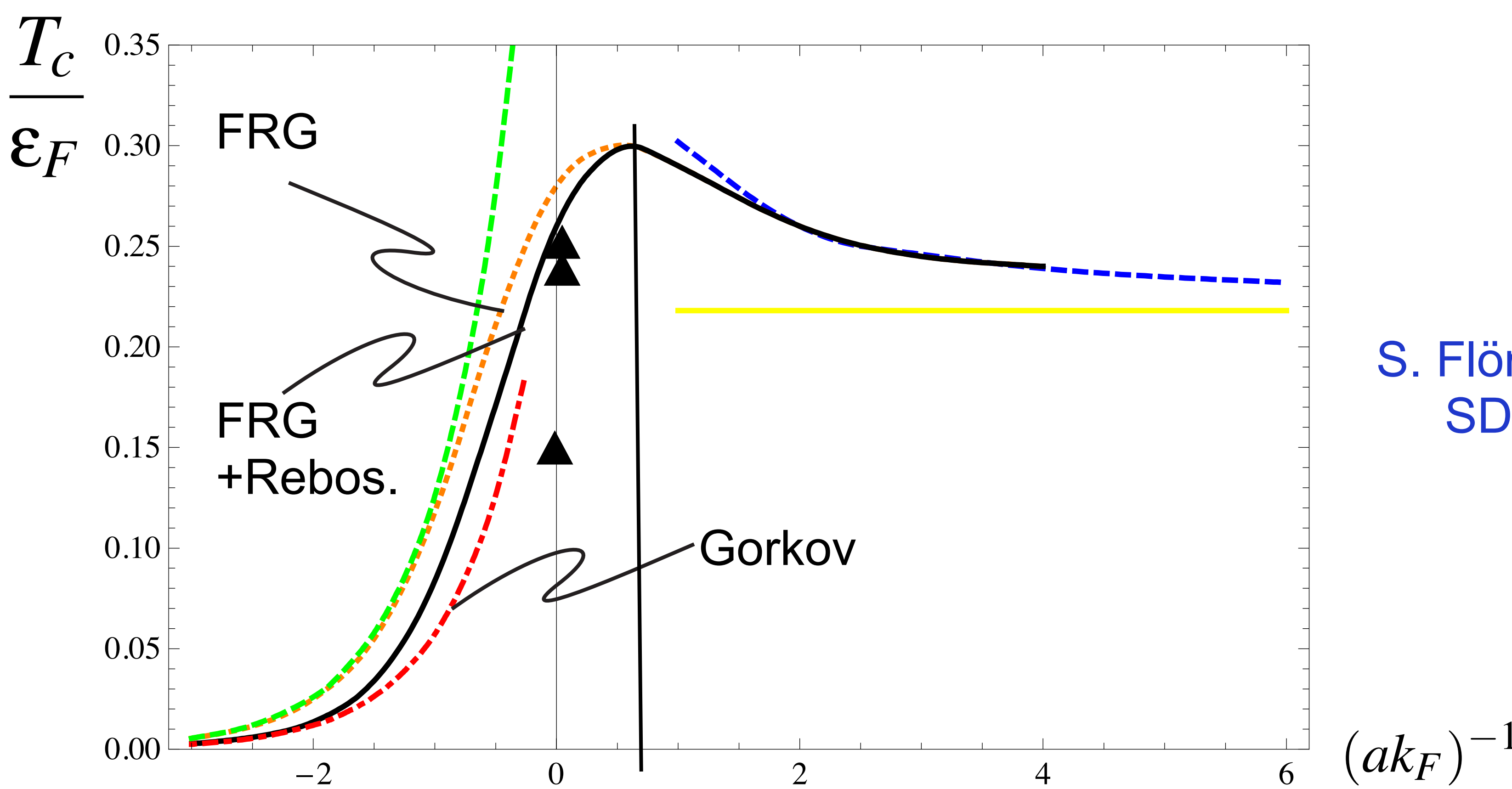
long distance

$$k_{ld} \gg n^{1/3}, T^{1/2}, \epsilon_M^{1/2}$$

Result: Particle-Hole Effects in the BCS-BEC crossover

microscopic	thermodynamic	long distance
$\varepsilon_M = -\frac{1}{Ma^2}$	$n = \frac{k_F^3}{3\pi^2}, T$	$k_{ld} \gg n^{1/3}, T^{1/2}, \varepsilon_M^{1/2}$

▲ QMC



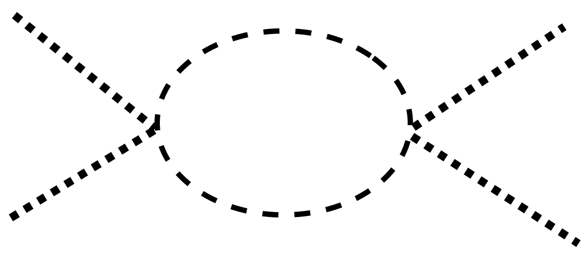
zero crossing of fermion chemical potential

S. Flörchinger, M. Scherer, SD, C. Wetterich ’08

- Accurately reproduce Gorkov effect in the BCS regime from rebosonization procedure: bosons massive even close to phase transition
- Fermion many-body effect: vanishes at zero crossing of chem. pot.
- but particle hole fluctuations are not the origin of the strong suppression of T_c wrt. simpler trunc.

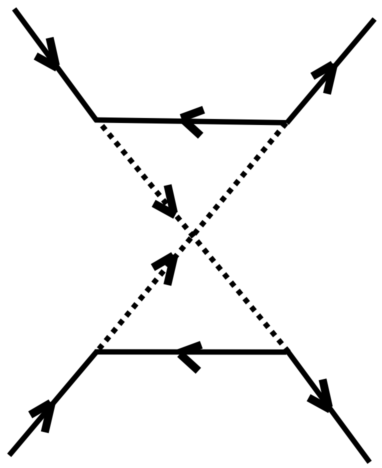
Renormalization of the Fermion Propagator

- So far: Interpolation scheme following evolution of beyond mean field effects into strongly interacting limit
  - boson particle-particle fluctuations



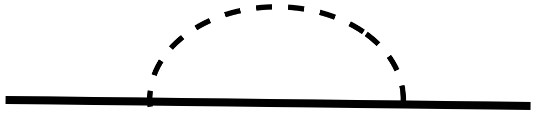
- drives (shift in T_c) on BEC side
- but bosons massive on BCS side (except very close phase transition): small effect in BCS regime

  - particle-hole fluctuations



- drives Gorkov effect on BCS side (perturbative Gorkov effect assumes massive bosons)
- but fermions massive on BEC side: small effect in BEC regime

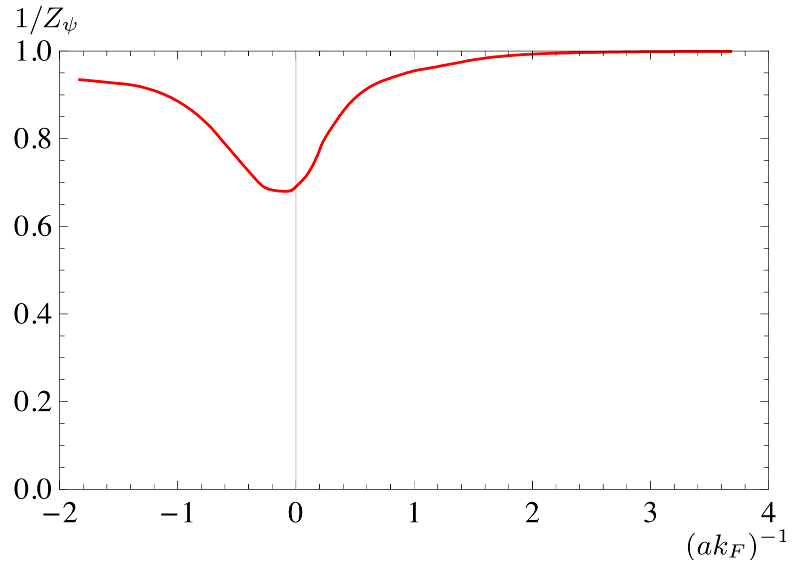
- Instead, renormalization effect on Fermion propagator strongest in crossover regime



- fermions massive in BEC regime
- bosons massive in BCS regime
- no obvious suppression in strongly int. regime

Result (S. Flörchinger, M. Scherer, C. Wetterich '10)

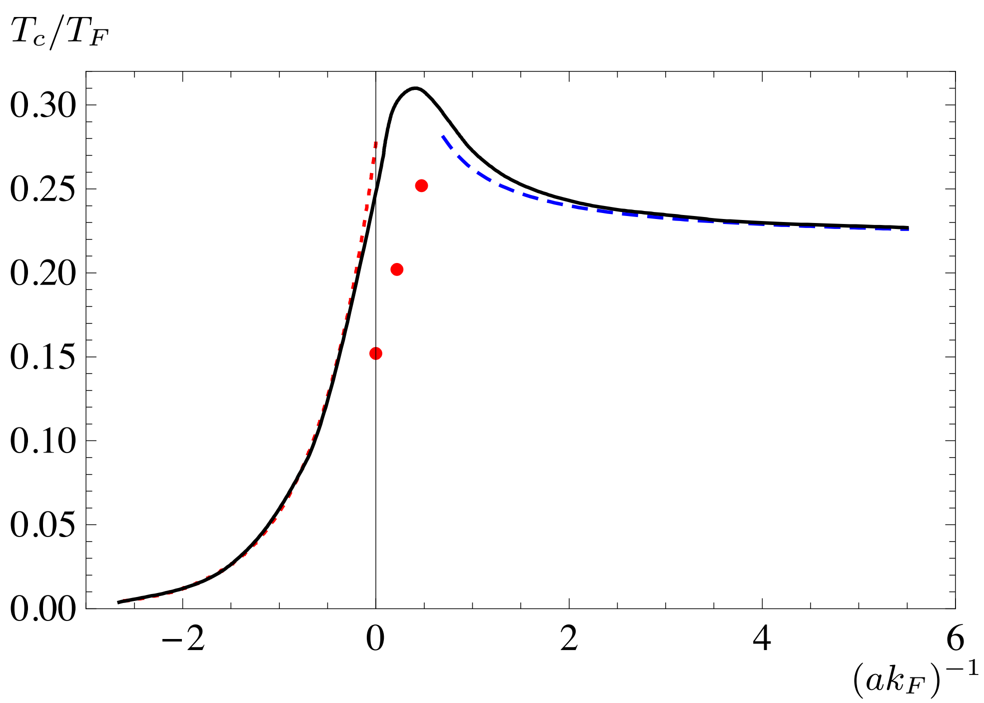
- Exemplarily, consider fermionic wave function renormalization $Z_\psi(T=T_c) = \frac{\partial}{\partial(\mathrm{i}\omega)}\Gamma^{(2)}_{\psi,k=0}$



- Numbers from most recent truncation compared to other approaches

at critical point and unitarity	μ_c/E_F	T_c/T_F
Burovski *et al.* (2006) (QMC)	0.49	0.15
Bulgac *et al.* (2006) (QMC)	0.43	< 0.15
Akkineni *et al.* (2007) (QMC)	-	0.245
Floerchinger *et al.* (2010) (FRG)	0.55	0.248

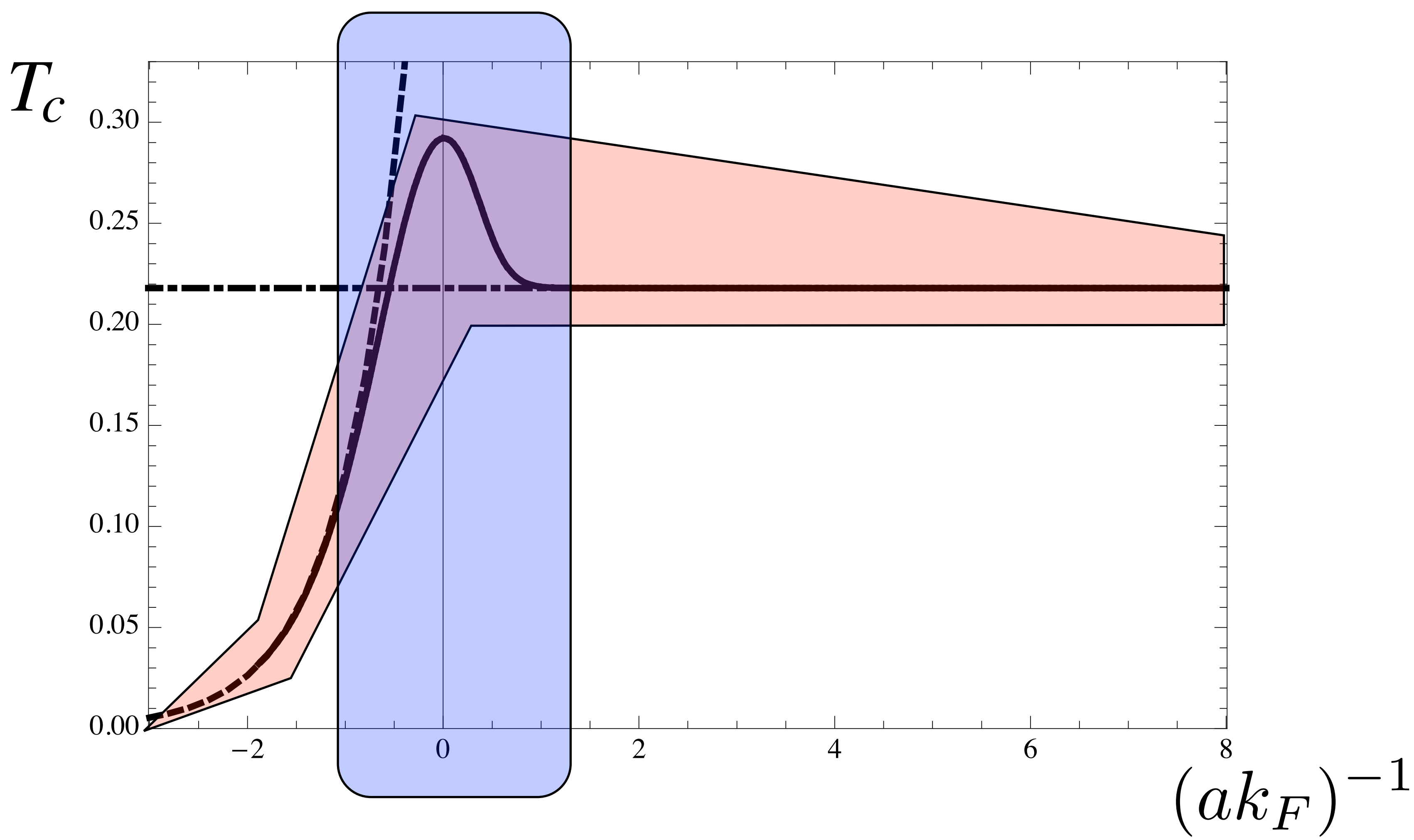
- Result for the phase diagram



at T = 0	μ/E_F	Δ/E_F
Carlson *et al.* (2003) (QMC)	0.43	0.54
Perali *et al.* (2004) (t-matrix approach)	0.46	0.53
Haussmann *et al.* (2007) (2PI)	0.36	0.46
Bartosch *et al.* (2009) (FRG, vertex exp.)	0.32	0.61
Floerchinger *et al.* (2010) (FRG, derivative exp.)	0.51	0.46

- convergence: minor quantitative change in T_c despite substantial renormalization of fermion propagator in strongly interacting regime

Aspects of Universality in the BCS-BEC Crossover



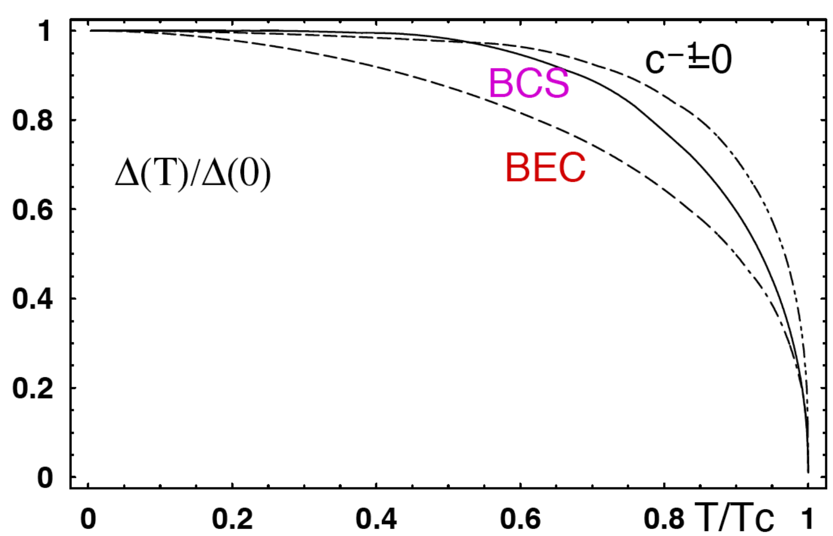
Universality I: Physics Close to the Phase Transition

SD, Gies, Pawlowski, Wetterich ’07; SD & ’10

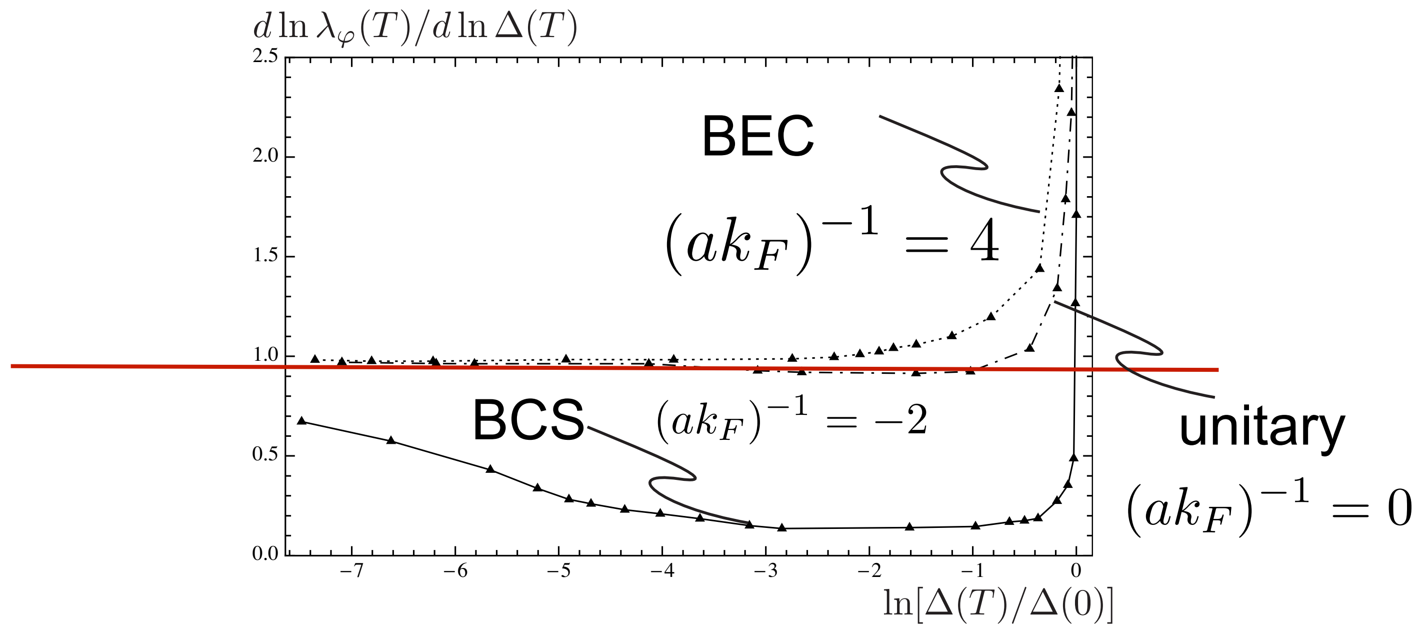
Close to (expected!) second order phase transition: Deep IR physics important

- fermion flow frozen out by temperature, i.e. wavelengths $k \sim \sqrt{T_c}$
- IR flow governed by Wilson-Fisher fixed point for d=3 O(2) model

Gap parameter in various regimes



Near-criticality: Extent of universal domain



- Second order PT throughout crossover (unlike e.g. Popov theory)
- continuous change of relevant dof.s
- Universal critical behavior of O(2) universality class from fermionic microscopic model:

$$\eta(1/(ak_F)) = 0.05 \quad \text{for all} \quad ak_F$$

- Investigate scaling of four-boson coupling on approaching the phase transition with $\Delta(T \to T_c)$

$$\lambda_\phi \sim \Delta^\zeta \qquad \zeta = 0.98$$

- largest critical domain in the unitary regime (fastest approach to scaling behavior)

microscopic

$\varepsilon_M = -\frac{1}{Ma^2}$

thermodynamic

$n = \frac{k_F^3}{3\pi^2}, T$

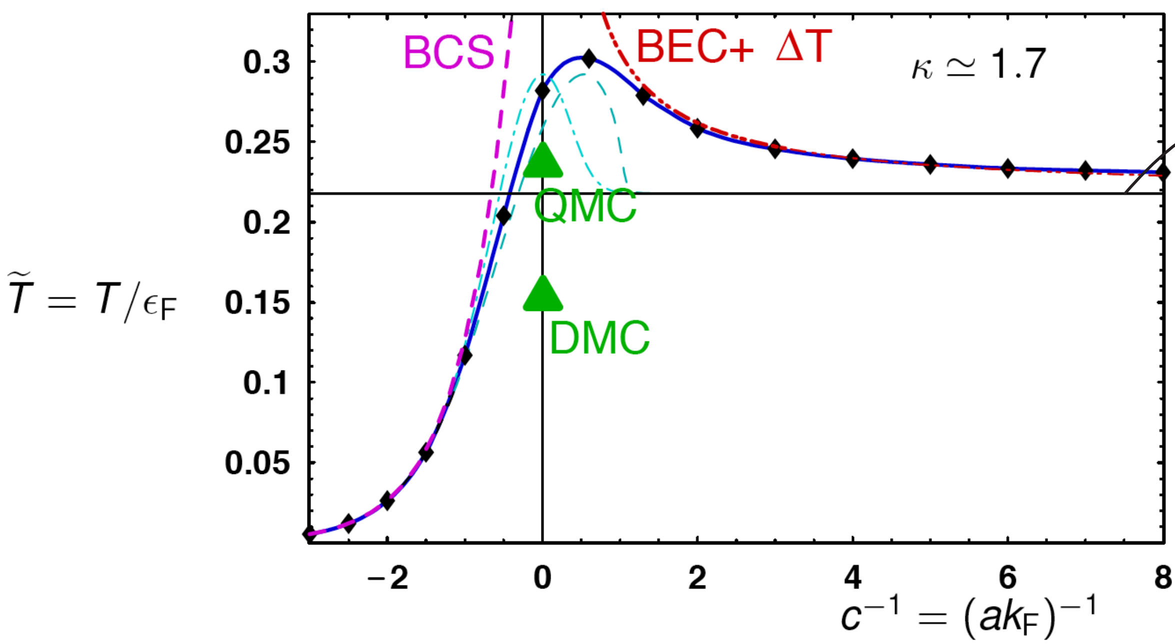
long distance

$k_{ld} \gg n^{1/3}, T^{1/2}, \varepsilon_M^{1/2}$

Universality I: Physics Close to the Phase Transition

- Manifestation of the quantitative control of physics on all scales is the calculation of the critical temperature in the BEC regime (SD, Gies, Pawlowski, Wetterich '07)

Phase Diagram



Free BEC critical Temperature

$$\boxed{\frac{T_c - T_c^{\mathrm{BEC}}}{T_c^{\mathrm{BEC}}} = \kappa \cdot a_M \cdot n^{1/3}}$$

- Microphysics: Scattering of composite bosons with effective molecular scattering length
- Thermodynamics / near critical behavior: Shift in the critical temperature (fund. bosons Baym, Blaizot '01)
- Critical behavior: d=3 O(2) universality class

	FRG	Other
$\frac{a_M}{a}$	0.75 (0.58) Birse& '10	0.6 Shlyapnikov & '04 Schrodinger Eq.
κ	1.7	1.3 Arnold& '01 lattice sim; Blaizot & '08 FRG
η	0.05	0.038 numerical simulations

microscopic → thermodynamic → long distance

$\epsilon_M = -\frac{1}{Ma^2}$ | $n = \frac{k_F^3}{3\pi^2}, T$ | $k_{ld} \gg n^{1/3}, T^{1/2}, \epsilon_M^{1/2}$

Universality II: Broad Resonance Universality

- Consider dependence of the effective action on the Feshbach coupling

$$\Gamma_k[\psi,\phi] = \int_0^{1/T} d\tau \int d^3x \Big\{ \psi^\dagger\big(\partial_\tau - \frac{\triangle}{2M} - \mu\big)\psi - \frac{h_{\phi,k}}{2}\big(\phi^*\psi^T\epsilon\psi - \phi\psi^\dagger\epsilon\psi^*\big)$$
$$+\phi^*\big(Z_{\phi,k}\partial_\tau - A_{\phi,k}\triangle + m^2_{\phi,k}\big)\phi + \lambda_{\phi,k}(\rho-\rho_0)^2 + \dots\Big\}$$

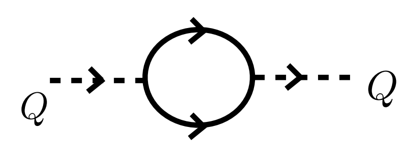
- The Feshbach coupling only renormalizes weakly, so $h_{\phi,k} \approx h_{\phi,in}$ for all k

- Classification
  - Broad resonances: $h_{\phi,in} \to \infty$
  - Narrow resonances: $h_{\phi,in} \to 0$

for fixed scattering length $a \sim \dfrac{h^2_{\phi,in}}{\nu(B)} = \text{const.}$

- Loop corrections scale with powers of $h_{\phi,in}$?!

e.g. inverse boson propagator



$\sim h^2_{\phi,in}$

➡ Redefine $\phi \to \tilde{\phi} = h_{\phi,k}\phi$

Universality II: Broad Resonance Universality

- Consider dependence of the effective action on the Feshbach coupling

$$\Gamma_k[\psi,\phi] = \int_0^{1/T} d\tau \int d^3x \Big\{ \psi^\dagger(\partial_\tau - \frac{\triangle}{2M} - \mu)\psi + \frac{1}{2}\Big(\tilde\phi^*\psi^T\epsilon\psi - \tilde\phi\psi^\dagger\epsilon\psi^*\Big)$$

$$\phi \to \tilde\phi = h_{\phi,k}\phi$$

$$+\tilde\phi^*\Big(\frac{Z_{\phi,k}}{h^2_{\phi,k}}\partial_\tau - \frac{A_{\phi,k}}{h^2_{\phi,k}}\triangle + \frac{m^2_{\phi,k}}{h^2_{\phi,k}}\Big)\tilde\phi + \frac{\lambda_{\phi,k}}{h^4_{\phi,k}}(\tilde\rho - \tilde\rho_0)^2 + \dots\Big\}$$

- Broad resonances: $h_{\phi,in} \to \infty$
- Narrow resonances: $h_{\phi,in} \to 0$

for fixed scattering length $a \sim \frac{h^2_{\phi,in}}{\nu(B)} = \mathrm{const.}$

- For broad resonances:
  - Initial conditions for most bosonic couplings do not matter for broad resonances: Universality!
  - yet there is one “relevant” coupling:

const. + loop corrections

$$\frac{m^2_{\phi,k}}{h^2_{\phi,k}} = \frac{\nu(B)}{h^2_{\phi,k}} + \dots \sim a^{-1} + \dots$$

- Thus:
  - For $1/a \to 0$ (Feshbach resonance): nonperturbative theory, as the dominant nonlinearity (cubic hbach term) is $\mathcal{O}(1)$
  - For $1/a \to \pm\infty$ (BCS/BEC regimes): ordering principle due to large bare boson mass

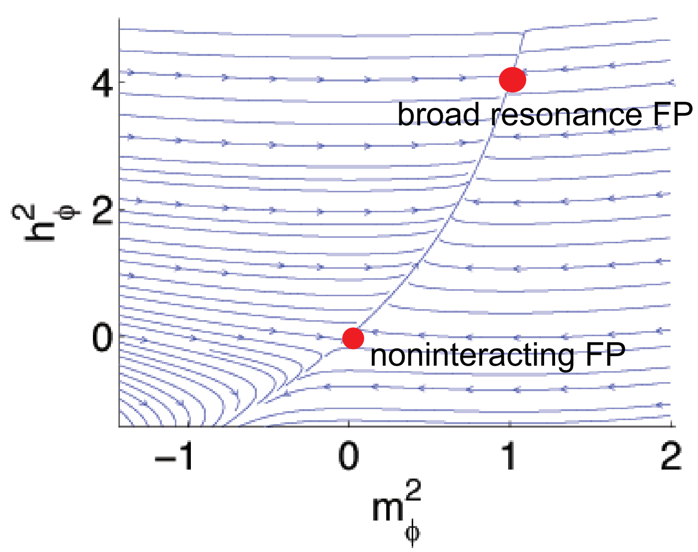
Broad vs. Narrow Resonances: RG perspective

(SD, Gies, Pawlowski, Wetterich ’07)

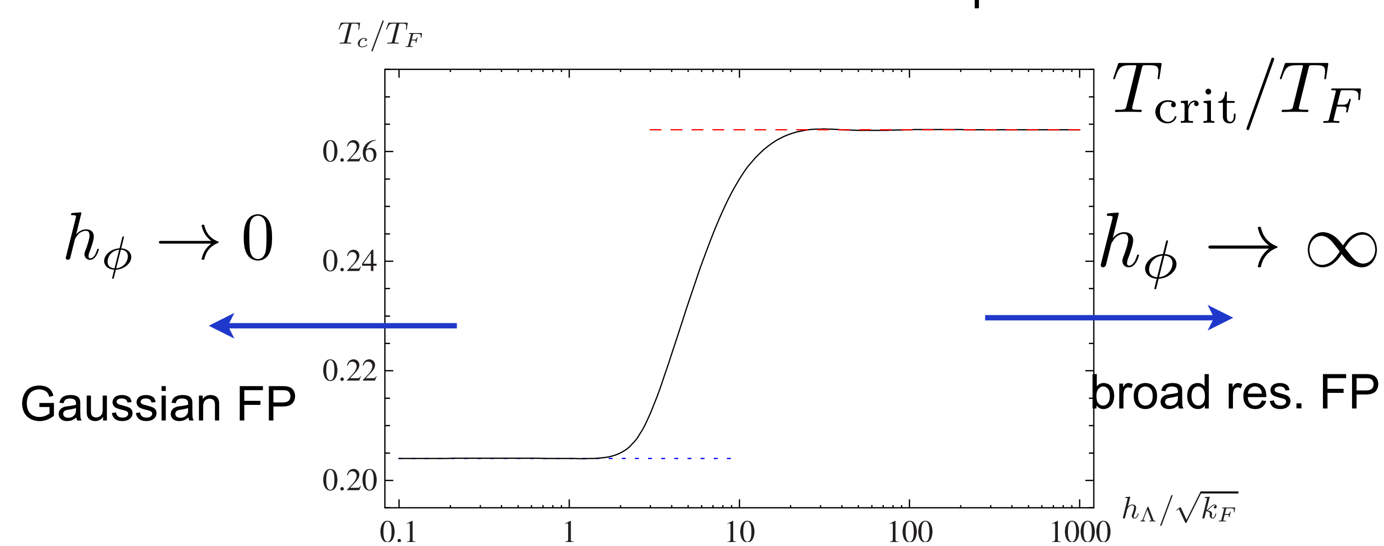
Identify two fixed points in model with detuning $\nu(B)\hat{=}m^2_{\phi,0}$ and Feshbach coupling $h_{\phi,0}$ (or $\{a^{-1}(B), h_{\phi,0}\}$):

- Broad resonances: Interacting FP
  - Detuning $\sim \frac{B-B_0}{B_0}$ single relevant perturbation: All further microscopic memory lost.
  - Similar to critical behavior near 2^{nd} order phase transition (single relevant perturbation $\sim \frac{T-T_c}{T_c}$)
- Narrow resonances: Gaussian FP
  - Detuning and Feshbach coupling relevant parameters: Microscopic information important.
  - Exact mean field-type solution available: minimally couple Bose-Fermi mixture which exhibits the full crossover behavior (SD, Wetterich ’05).

Fixed point structure



Narrow to broad resonance crossover: dominance of different fixed points

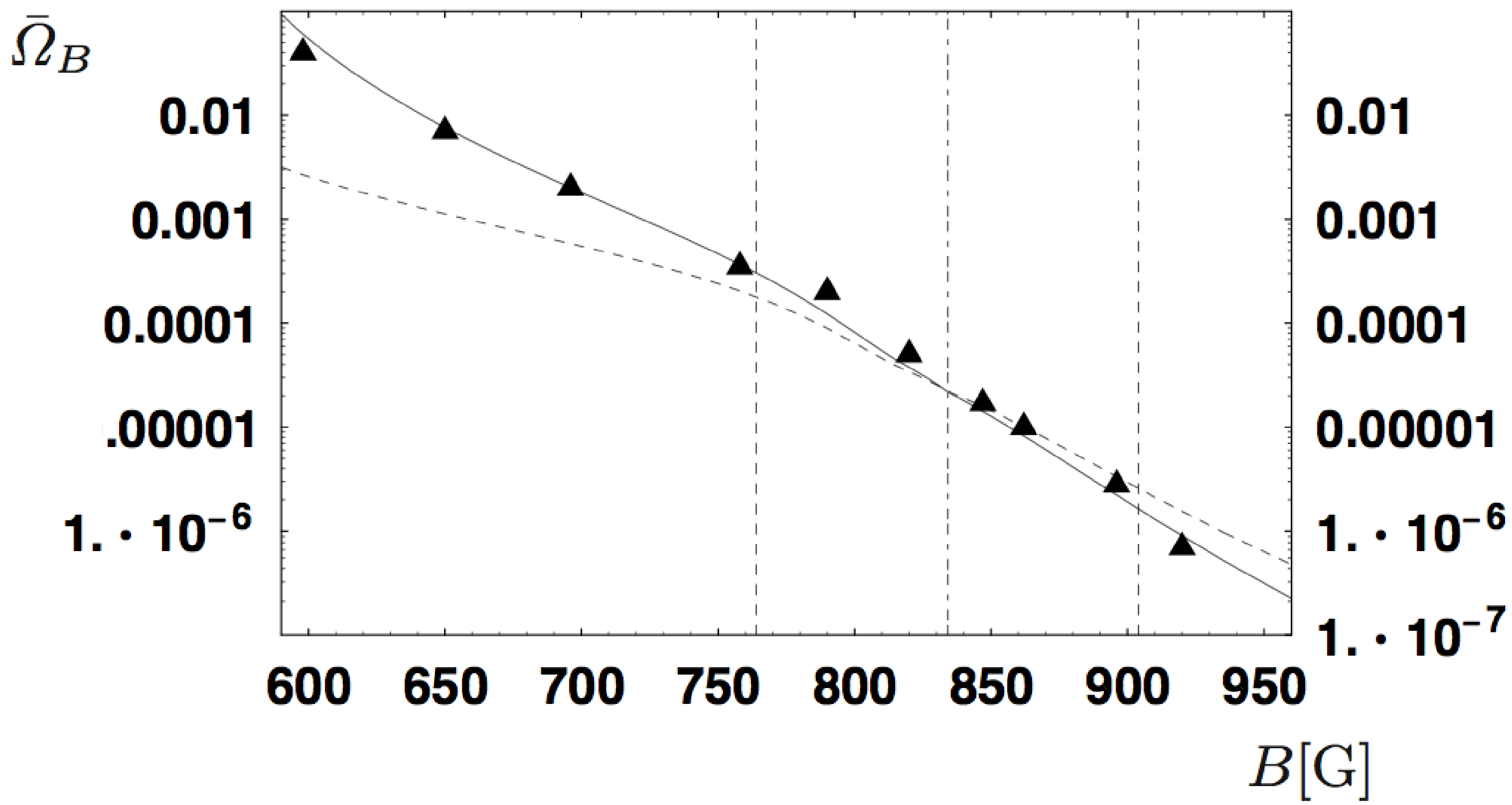


➡ Explains universality in crossover experiments (K,Li atoms) from RG point of view

➡ Further possibility for perturbative expansion about narrow resonance FP (cf. epsilon, 1/N expansions (Nishida, Son '05; Radzihovsky, Sheey '06; Nikolic, Sachdev '06)

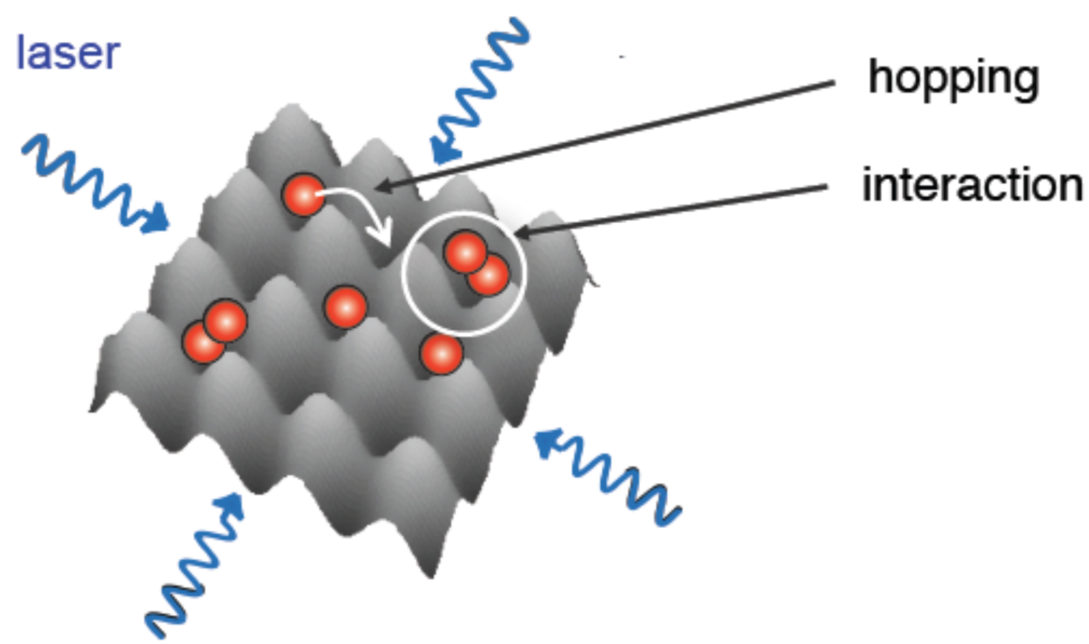
Scaling Violations in Crossover Experiments

- Large but finite Feshbach coupling induces scaling violations in many-body system.
- Deviations from universality probed experimentally Partridge et al. '05
- Measure the closed channel population probability $\bar{\Omega}_B$.
- Scaling violation $\mathcal{O}(k_F h_{\phi,0}^{-2})$



(SD, Wetterich '05)

Strongly Correlated Bosons: The Bose-Hubbard Model



cold atoms in an optical lattice (see below)

Microscopic Origin: Bosons in Periodic Optical Potentials

- Starting point: workhorse Hamiltonian for weakly interacting ultracold bosons

$$H = \int_{\mathbf{x}} [\hat\psi^\dagger_{\mathbf{x}}(-\frac{\triangle}{2m} + V(\mathbf{x}))\hat\psi_{\mathbf{x}} + g(\hat\psi^\dagger_{\mathbf{x}}\hat\psi_{\mathbf{x}})^2]$$

- see above: trapping potential can be treated classically due to scale separation
- instead, now we are interested in a periodic potential of wavelength comparable to the typical interparticle distance: light in with optical wavelength, as

$$\lambda \sim 500\mathrm{nm} = 5\cdot 10^{-5}\mathrm{cm}$$

typical wavelength of light

$$d \lesssim 10^{-4}\mathrm{cm} \quad (n \gtrsim 10^{12}\mathrm{cm}^{-3})$$

typical interparticle separation

- create such conservative potential by weakly coupling the atoms in their ground state $(\leftrightarrow \hat\psi_{\mathbf{x}})$ to auxiliary internal level: position dependent second order AC Stark shift for standing wave laser beam yields optical potential

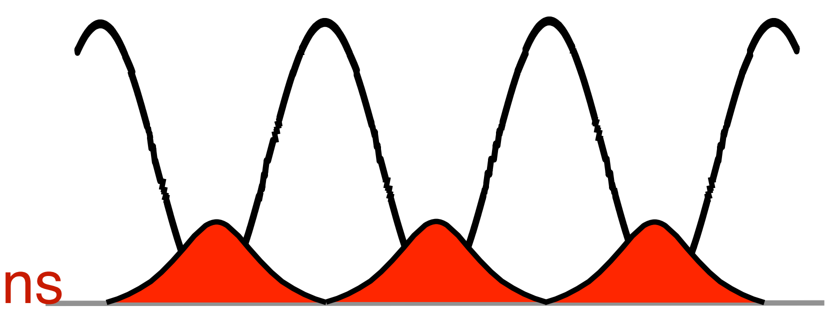
Rabi coupling to aux. level

$$V(\mathbf{x}) = \hbar\frac{\Omega^2(\mathbf{x})}{4\Delta} \equiv V_0\sum_i \sin^2(k_i x_i), \quad k_i = 2\pi/\lambda_i$$

laser detuning from aux. level

- atomic fermions can be treated analogously: Fermi Hubbard model

Bose Hubbard Hamiltonian (Jaksch et al. '98)

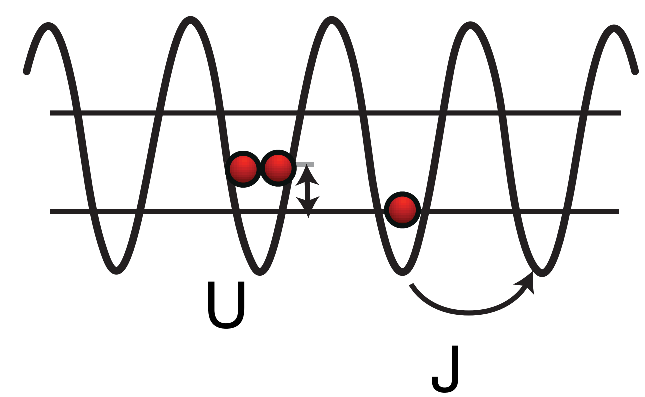


- For dominant optical potential $V_0 \gg$(other scales), we expect localized single particle wavefunctions to provide a useful description of the system.

- A suitable complete set of basis functions are the Wannier functions

- We expand the field operators in Wannier functions of the lowest band

$$\hat{\psi}(\vec{x}) = \sum_i w(\vec{x} - \vec{x}_i) b_i$$



to obtain the Bose Hubbard model

$$\boxed{\hat{H} \quad = \quad -\sum_{ij} J_{ij} b_i^\dagger b_j + \tfrac{1}{2} U \sum_i b_i^{\dagger 2} b_i^2}$$

with hopping $J_{ij} = \int d^3x w(\vec{x} - \vec{x}_i) \left[-\frac{\hbar^2}{2m} \nabla^2 + V_0(\vec{x}) \right] w(\vec{x} - \vec{x}_j)$ and interaction $U = \frac{1}{2} g \int d^3x \, |w(\vec{x})|^4$ valid for $J, U, k_B T \ll \hbar\omega_{\text{Bloch}}$. (tight binding lowest band approximation)

---

additionally, we are bound to interactions (scattering lengths)

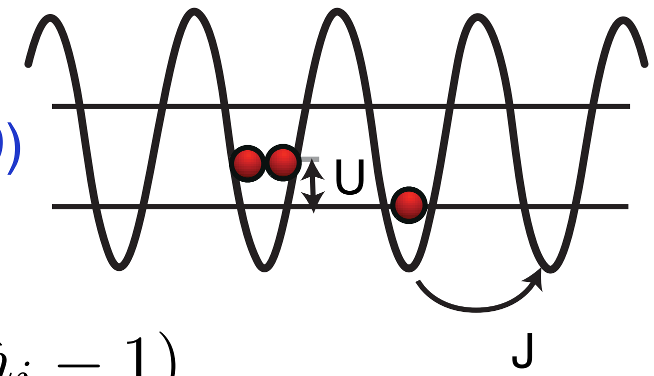
extent of Wannier function

$$g \ll a_0, a$$

here, it means lattice spacing

This is not true close to Feshbach resonances!

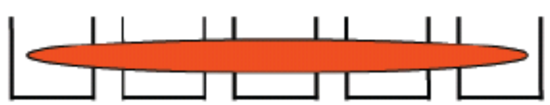
Bose-Hubbard Model (Fisher, Weichman, Grinstein, Fisher '89)



$$H = \underbrace{-J \sum_{\langle i,j \rangle} b_i^\dagger b_j}_{\text{kinetic energy}} - \mu \sum_i \hat{n}_i + \underbrace{\sum_i \epsilon_i \hat{n}_i}_{\substack{\text{trapping potential} \\ \text{(ignored here)}}} + \underbrace{\tfrac{1}{2} U \sum_i \hat{n}_i(\hat{n}_i - 1)}_{\text{interaction energy}}$$

- Lattice model: Possible to penetrate high density regime $\langle \hat{n}_i \rangle = \mathcal{O}(1)$. Not possible in continuum.
- Ratio of kinetic and interaction energy tunable via lattice parameters (and Feshbach resonances). In particular, reach interaction dominated regime.
- The competition of kinetic and interaction energy gives rise to a quantum phase transition

kinetic energy dominates: superfluid



interaction energy dominates: "Mott insulator"

$T = 0$ $\longrightarrow$ U/J

"strongly correlated"

$U/J \gg 1$

- The Bose-Hubbard model is an exemplary model for strongly correlated bosons. It is not realized in condensed matter.

- Remark: strong interactions and high density not in contradiction to earlier scale considerations:
  - strong interactions: $J/U \ll 1$ mainly from reduction of kinetic energy via lattice depth.
  - High density due to strong localization of onsite wave function.
  - For validity of lowest band approximation, it is however important that $a \ll \lambda$

What is a Quantum Phase Transition (QPT)?

- Definition: A phase transition at T=0 which results from two competing (noncommuting) operators in a Hamiltonian, each of which prefers ground state with different symmetry
- Second order QPT are characterized by spatial and temporal critical exponent
  - diverging length scale describing the decay of spatial correlations at the transition point

critical exponent

$$\xi^{-1} \sim |g - g_c|^{\nu}$$

  - vanishing energy scale separating ground from excited states (gap) at the transition point implies diverging time scale

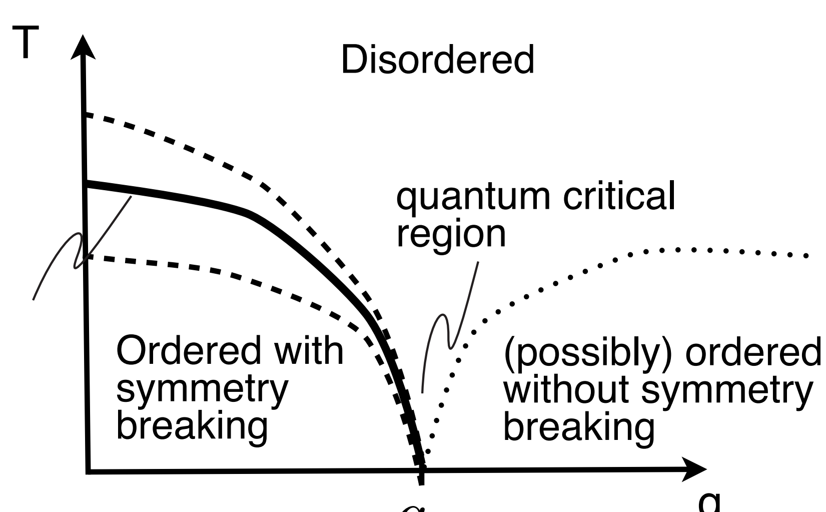
$$\Delta \sim |g - g_c|^{\nu z_d}$$

  - the ratio defines the dynamic critical exponent,

$$\Delta \sim \xi^{-z_d}$$

- A finite temperature is always a relevant perturbation at the quantum critical point. Therefore, a generic “quantum phase diagram” has a shape

line of second order phase transitions



eg. BH model $g = U/J$: superfuid Mott insulator

- classical description of critical behavior applies if

$$\hbar\omega_{typ} \ll k_B T$$

- This is always violated at low enough T: classical-quantum crossover

Mean Field Phase Diagram: Strong Coupling Expansion

- On the lattice, the strong coupling limit is simple and exactly solvable: J = 0 corresponds to an array of decoupled sites

$$H = -J\sum_{\langle i,j\rangle} b_i^\dagger b_j \underbrace{- \mu \sum_i \hat{n}_i + \tfrac{1}{2}U\sum_i \hat{n}_i(\hat{n}_i - 1)}_{\text{diagonal in Fock space, exactly solvable}}$$

- Mean field theory via

  local condensate mean field

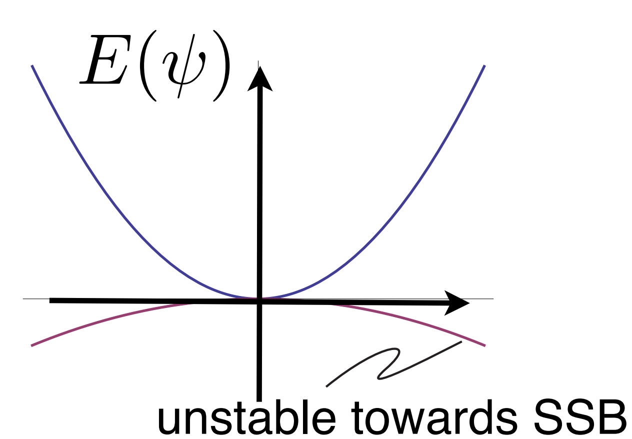
  - Ansatz: $b_i = \psi + \delta b_i$
  - Insert into H and rewrite: $H = H^{(\mathrm{MF})} + \cancel{\sum_{i,j} \mathcal{O}(\delta b_i^\dagger \delta b_j)}$
  - with local mean field Hamiltonian $H^{(\mathrm{MF})} = \sum_i h_i$ expressed in orig. operators again

$$h_i = -\mu\hat{n}_i + \tfrac{1}{2}U\hat{n}_i(\hat{n}_i - 1) - Jz(\psi^* b_i + \psi b_i^\dagger) + Jz\psi^*\psi$$

  - Assume second order phase transition and follow Landau procedure:
    - Study ground state energy

$$E(\psi) = \mathrm{const.} + m^2|\psi|^2 + \mathcal{O}(|\psi|^4)$$

    - Determine zero crossing of mass term in second order perturbation theory in $J\psi \ll 1$ close to phase transition



Phase Diagram: Derivation

additional material

- Assume second order phase transition and follow Landau procedure:
  - Study ground state energy

$$E(\psi) = \text{const.} + m^2|\psi|^2 + \mathcal{O}(|\psi|^4)$$

  - Determine zero crossing of mass term

- Calculate E in second order perturbation theory

$$h_i = h_i^{(0)} + \psi V_i$$

smallness parameter close to phase transition

$$h_i^{(0)} = -\mu \hat{n}_i + \tfrac{1}{2} U \hat{n}_i(\hat{n}_i - 1) + Jz\psi^*\psi$$

$$V_i = Jz(b_i + b_i^\dagger)$$

- Validity: approximation neglects spatial correlations via local form
  - becomes exact in infinite dimensions (Metzner and Vollhardt ’89)
  - reasonable in d=2,3 (T=0)

Phase Diagram: Derivation

- Zero order Hamiltonian $h_i^{(0)}$: diagonal in Fock basis $\{|n\rangle\}, n = 0, 1, 2, ...$
- The eigenvalues are $E_n^{(0)} = -\mu n + \frac{1}{2}Un(n-1) + Jz\psi^2$
- The ground state energies for given μ are

$$E_{\bar{n}}^{(0)} = \begin{cases} 0 & \text{for } \mu < 0 \\ -\mu\bar{n} + \frac{1}{2}U\bar{n}(\bar{n}-1) + Jz\psi^2 & \text{for } U(\bar{n}-1) < \mu < U\bar{n} \end{cases}$$

- The second order correction to the energy is

$$E_{\bar{n}}^{(2)} = \psi^2 \sum_{n \neq g} \frac{|\langle \bar{n}|V_i|n\rangle|^2}{E_{\bar{n}}^{(0)} - E_n^{(0)}} = (Jz\psi)^2 \left(\frac{\bar{n}}{U(\bar{n}-1) - \mu} + \frac{\bar{n}+1}{\mu - U\bar{n}} \right)$$

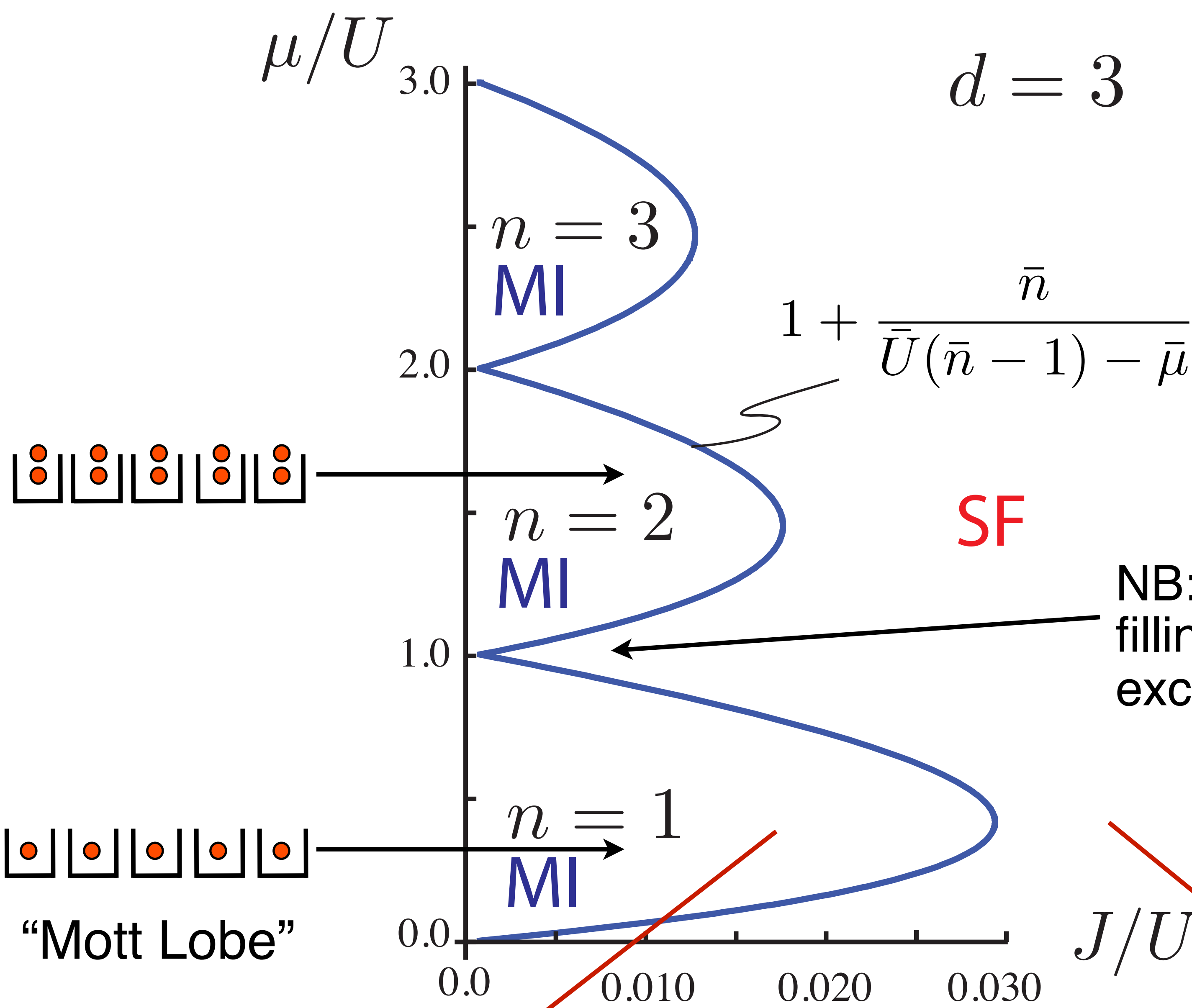
- For $E =$ const. $+ m^2\psi^2 + ...$ the phase transition happens at ($\bar{\mu} = \mu/Jz, \bar{U} = U/Jz$)

$$\frac{m^2}{Jz} = \boxed{1 + \frac{\bar{n}}{\bar{U}(\bar{n}-1) - \bar{\mu}} + \frac{\bar{n}+1}{\bar{\mu} - \bar{U}\bar{n}} = 0}$$

Bose-Hubbard mean field
phase border

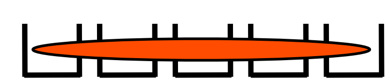
Phase Diagram: Overall Shape

This gives the phase diagram as a function of μ/U and J/U.



$$1 + \frac{\bar{n}}{\bar{U}(\bar{n}-1) - \bar{\mu}} + \frac{\bar{n}+1}{\bar{\mu} - \bar{U}\bar{n}} = 0$$

NB: for non-commensurate (= integer) fillings, superfluidity persists for U -> 0: excess particles condense.



Simple picture:

MI: Quantization of particle number

SF: Quantization of phase

$$\text{"}[\hat{N}, \hat{\varphi}] = \mathrm{i}\text{"}$$ - conjugate variables

The Mott Phase

- In the kinetically dominated limit, we expect a weakly correlated superfluid (see above)
- Here we discuss characteristic features of the Mott insulator
  - Mean field Mott state : $|\bar{n}\rangle = \prod_i |\bar{n}_i\rangle = \bar{n}^{-M/2} \prod_i b_i^{\dagger \bar{n}} |\text{vac}\rangle$: Quantization of particle number

- Quantization of particle number within MI is an exact result in the sense $\langle b_i^\dagger b_i \rangle = \bar{n}$
  - $*$ at $J = 0$, Mott state $|\bar{n}\rangle$ is (i) exact ground state, (ii) eigenstate to particle number $\hat{N} = \sum_i \hat{n}_i$, (iii) separated from other states by gap $\sim U$
  - $*$ kinetic perturbation $H_{\text{kin}} = -J \sum_{\langle i,j \rangle} b_i^\dagger b_j$
  - $*$ commutes with $\hat{N}$, $[H_{\text{kin}}, \hat{N}] = 0$

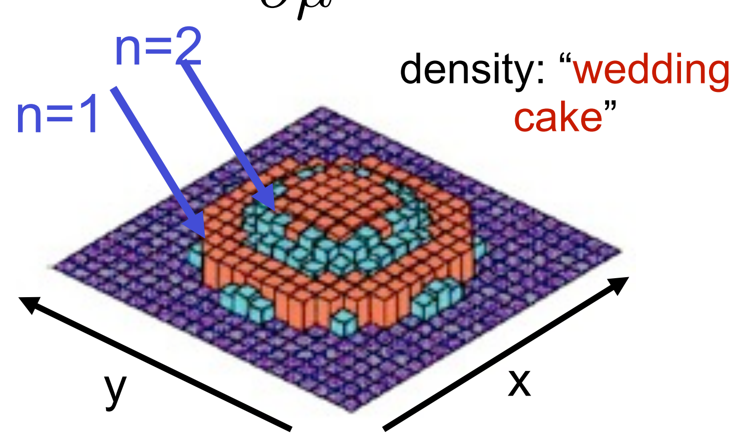
  ➡ switching on J adiabatically, the ground state remains exact eigenstate to number operator. Assuming translation invariance gives exact result on quantized particle number,

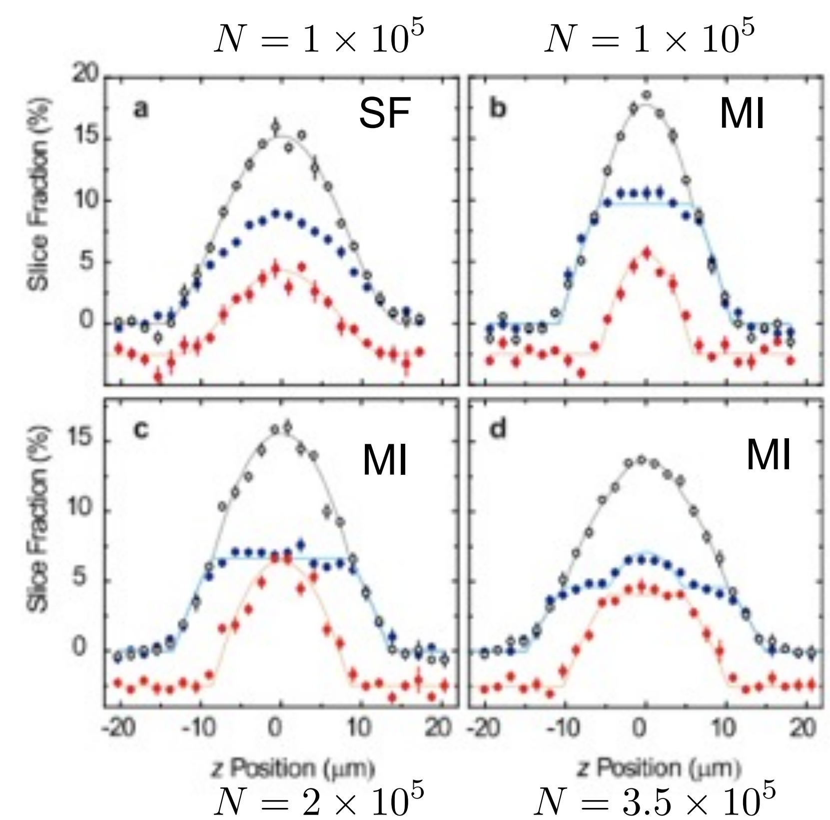
$$\langle b_i^\dagger b_i \rangle = \bar{n}$$

- Implication: the Mott insulator is a gapped incompressible state,

$$\frac{\partial \langle \hat{N} \rangle}{\partial \mu} = 0$$

- Observable consequence: wedding cake density profile





Bloch group, 2006

Functional RG approach

This part of the lecture is based on work by A. Rancon & N. Dupuis, arxiv: 1012.0166

- Idea:
  - Strong coupling mean field provides correct qualitative behavior of short distance physics and thermodynamics (phase diagram)
  - Use mean field theory as a starting point and include fluctuations via FRG equation
  - in this way, include both relevant short distance lattice physic and interpolate directly to long distances: “physics on all scales”

- Implementation: start from regularized Bose-Hubbard action:

$$S_k = S_{BH} + \Delta S_k \qquad S_{BH} = S_{\text{loc}} + S_{\text{kin}}$$

$$S_{\text{loc}} = \int d\tau \sum_i \varphi_i^* (\partial_\tau - \mu)\varphi_i + \frac{U}{2}(\varphi_i^* \varphi_i)^2$$

$$S_{\text{kin}} = -t \int d\tau \sum_{i,j} \varphi_i^* \varphi_j + c.c. = \int d\tau \sum_{\mathbf{q}} \varphi_{\mathbf{q}}^* t_{\mathbf{q}} \varphi_{\mathbf{q}}^*$$

$$t_{\mathbf{q}} = -2t \sum_{i=1}^{d} \cos q_i$$

bare lattice dispersion

Implementation: Cutoff Function

- Choice of the cutoff function:

$$\Delta S_k = \int d\tau \sum_{\mathbf{q}} \varphi^*_{\mathbf{q}} R_k(\mathbf{q}) \varphi_{\mathbf{q}} \qquad \text{cf.} \qquad S_{\text{kin}} = \int d\tau \sum_{\mathbf{q}} \varphi^*_{\mathbf{q}} t_{\mathbf{q}} \varphi_{\mathbf{q}}$$

$$R_k(\mathbf{q}) = -Z_{A,k} t k^2 \operatorname{sgn}(t_{\mathbf{q}})(1-y_{\mathbf{q}})\theta(1-y_{\mathbf{q}}) \qquad y_{\mathbf{q}} = 1 - (2dt - |t_{\mathbf{q}}|)/tk^2$$

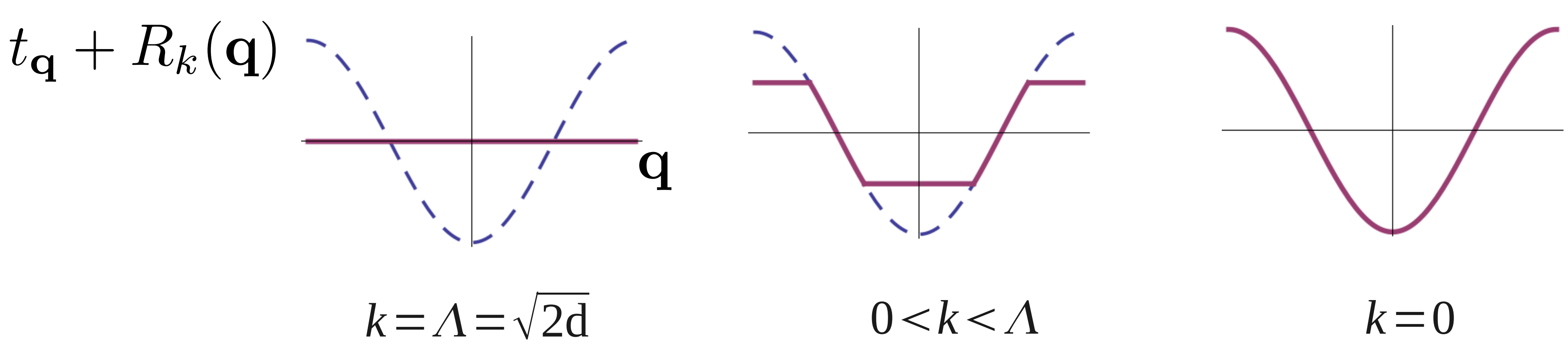
see truncation

- Limiting cases:

$$k = \Lambda : \quad t_{\mathbf{q}} + R_k(\mathbf{q}) = 0$$ action becomes local

$$k = 0 : \quad t_{\mathbf{q}} + R_k(\mathbf{q}) = t_{\mathbf{q}}$$ the full hopping is taken into account

- I.e. spatial fluctuations are suppressed in UV and fully present in IR



(courtesy N. Dupuis)

Initial Condition: Mean Field Theory

- Remember: effective running action is modified Legendre transform:

$$\Gamma_k[\phi^*, \phi] = -\log Z_k[J^*, J] + \int d\tau \sum_i (J_i^* \phi_i + c.c.) - \Delta S_k[\phi^*, \phi]$$

- The initial condition for the flow:

$$k = \Lambda : \quad \Gamma_\Lambda[\phi^*, \phi] = \Gamma_{\rm loc}[\phi^*, \phi] + \int d\tau \sum_{\mathbf{q}} \phi_{\mathbf{q}} t_{\mathbf{q}} \phi_{\mathbf{q}}$$

with

$$\Gamma_{\rm loc}[\phi^*, \phi] = -\log Z_\Lambda[J^*, J] + \int d\tau \sum_i (J_i^* \phi_i + c.c.)$$

standard Legendre transform of a local partition function

  - numerically exactly solvable local problem (for any temperature)
  - Γ_Λ is equivalent to the above mean field approximation, as $\phi_{\mathbf{q}}$ is the classical field

- Thus, the Wetterich flow equation will interpolate between the mean field approximation and the full theory reached at $\Gamma_{k\to 0}$

Truncation

- Having built in the short range correlations, we are now interested in thermodynamics and long wavelength physics
- Derivative expansion as for weakly interacting bosons (Wetterich & ’08; Kopietz & ’09; Dupuis ’09)

$$\Gamma_k^{(2)} = \begin{pmatrix} V_{A,k}\omega^2 + Z_{A,k}\epsilon_{\mathbf{q}} + V_k' + 2\rho V_k'' & -Z_{C,k}\omega \\ Z_{C,k}\omega & V_{A,k}\omega^2 + Z_{A,k}\epsilon_{\mathbf{q}} + V_k' \end{pmatrix}$$

with suitably normalized lattice dispersion

$$\epsilon_{\mathbf{q}} = t_{\mathbf{q}} + 2dt \approx t\mathbf{q}^2 \text{ for } \mathbf{q} \to 0$$

crossover to relativistic model at low energies (Wetterich ’08; Kopietz & ’09,’10; Dupuis ’09)

- Keep the full effective potential for the thermodynamics. For simplified discussion of long wavelength physics,

$$V_k(\rho) = \begin{cases} V_{0,k} + \frac{\lambda_k}{2}(\rho - \rho_{0,k})^2 & \text{for } \rho_{0,k} > 0 \quad \text{SSB} \\ V_{0,k} + \Delta_k\rho + \frac{\lambda_k}{2}\rho^2 & \text{for } \rho_{0,k} = 0 \quad \text{SYM} \end{cases}$$

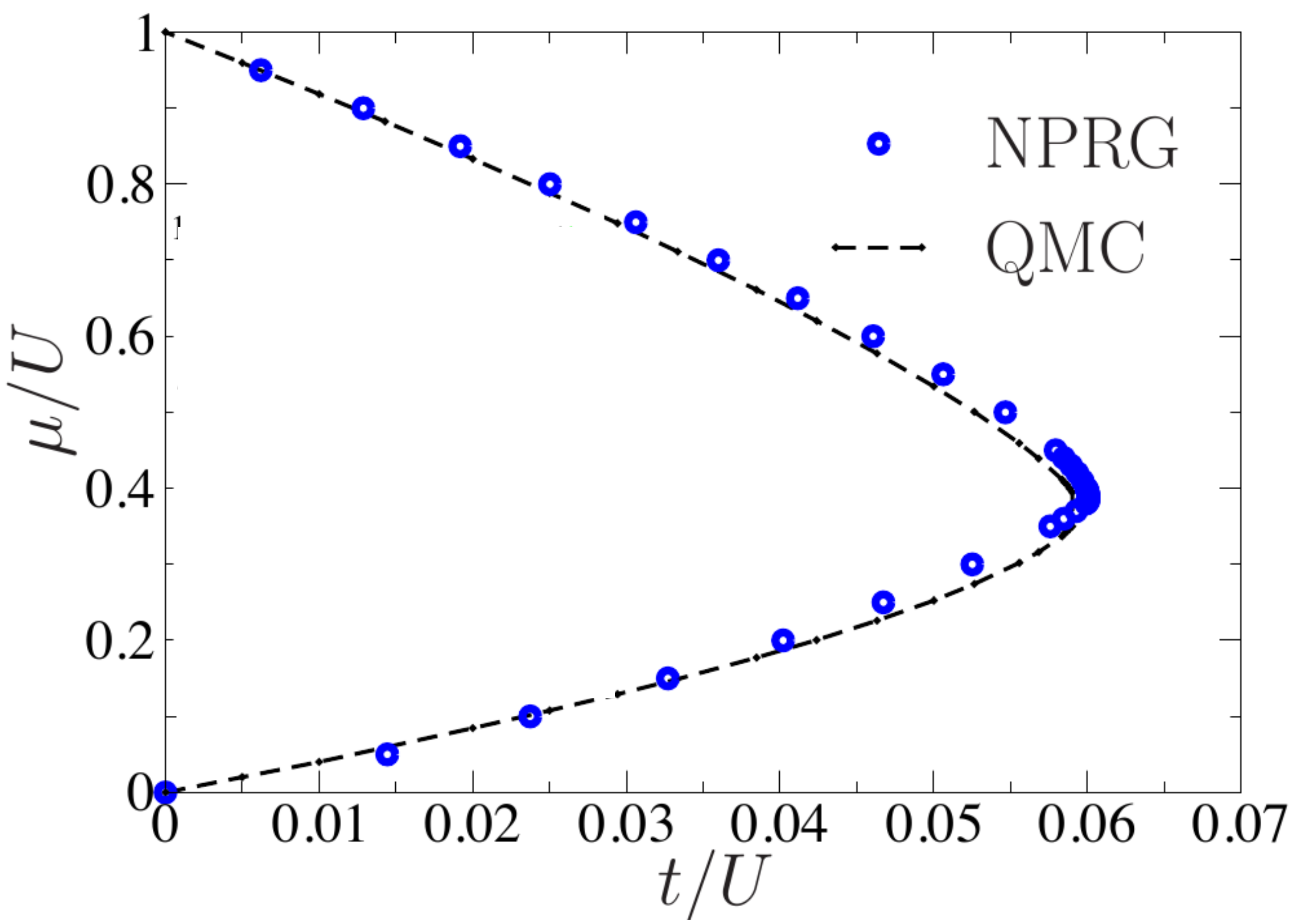
- As above, the average particle number can be obtained from the effective potential

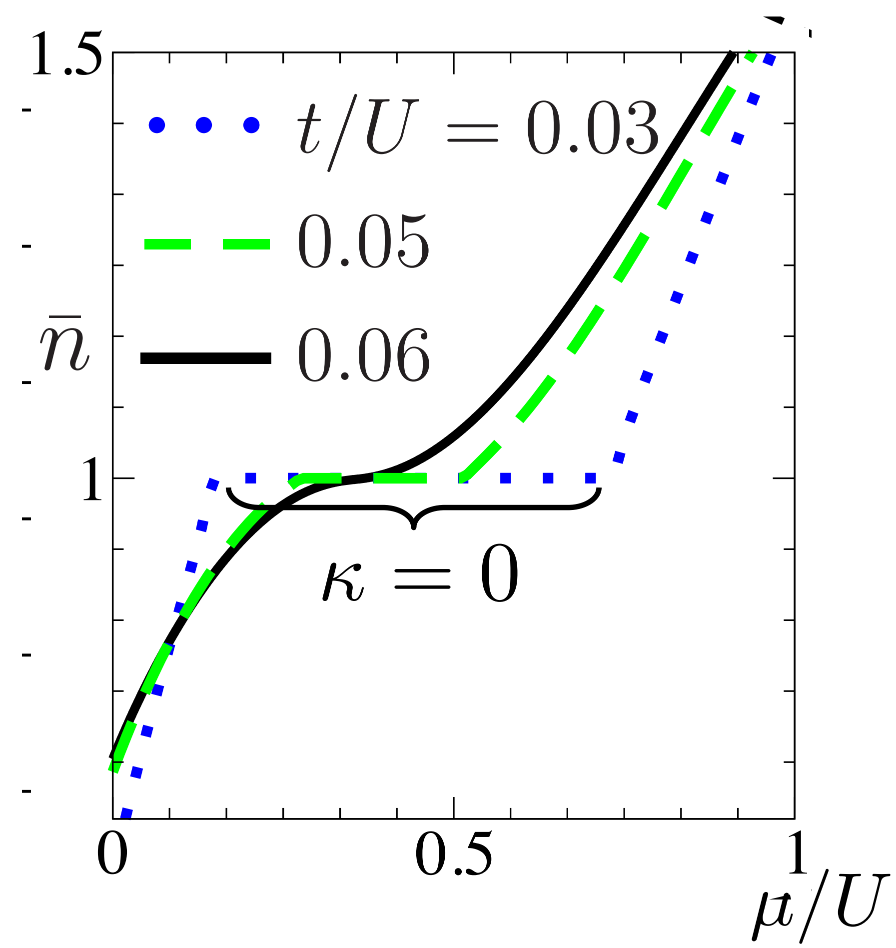
$$\bar{n} = -\frac{\partial V_{0,k\to 0}}{\partial \mu}$$

Phase Diagram and Thermodynamics

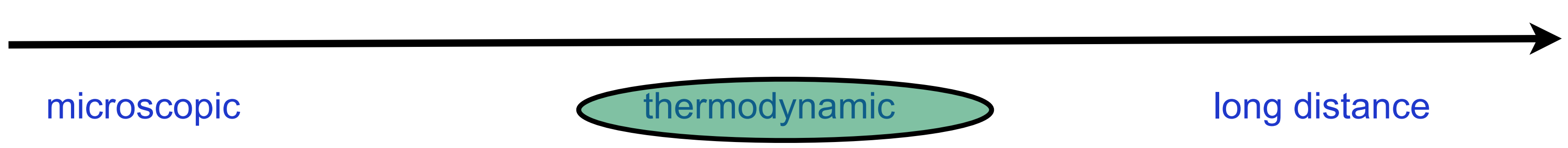
$$\kappa = \bar{n}^2 \frac{\partial \bar{n}}{\partial_\mu}$$

- Good agreement with recent QMC simulations for the phaser border (cf. MFT tip of the lobe: ca. 20% deviation)
- Compressibility shows plateau behavior associated to particle number quantization





(A. Rancon & N. Dupuis, arxiv:1012.0166
QMC Data: B. Capogrosso-Sansone ’08)



A Strongly Correlated Superfluid

- Remember: two scales in a superfluid (see weakly interacting bosons): use d=2 and, at low quasimomenta, $2m \approx 1/t$

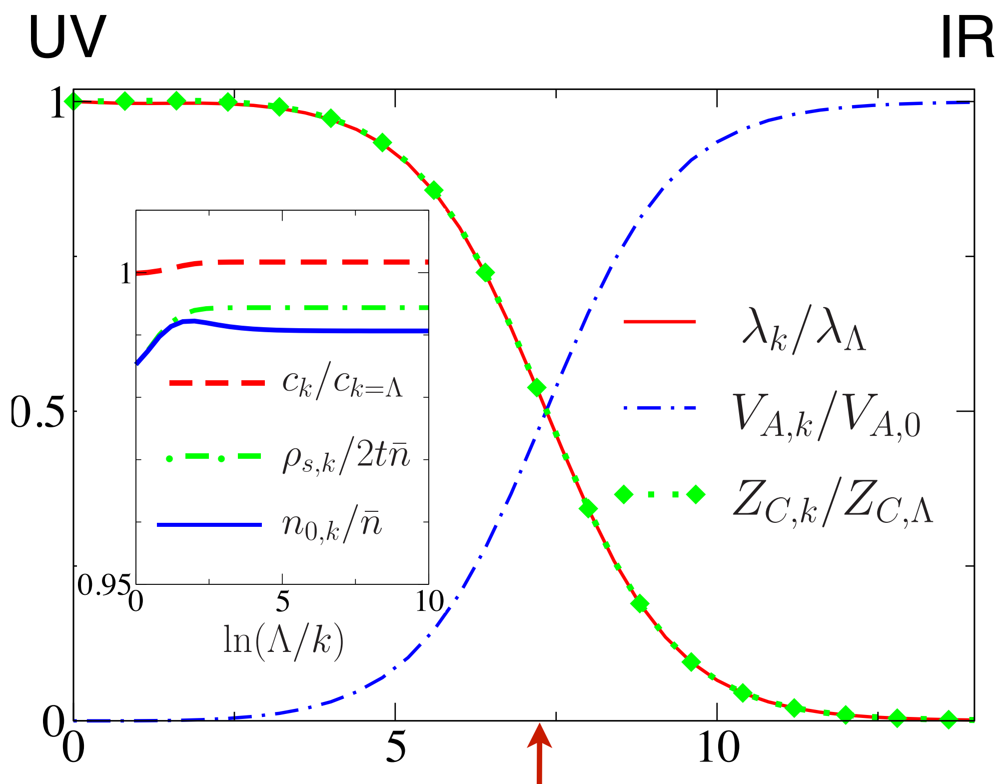
$$p_h = \sqrt{\bar{n}(U/t)}$$

crossover from quadratic to linear (Bogoliubov)

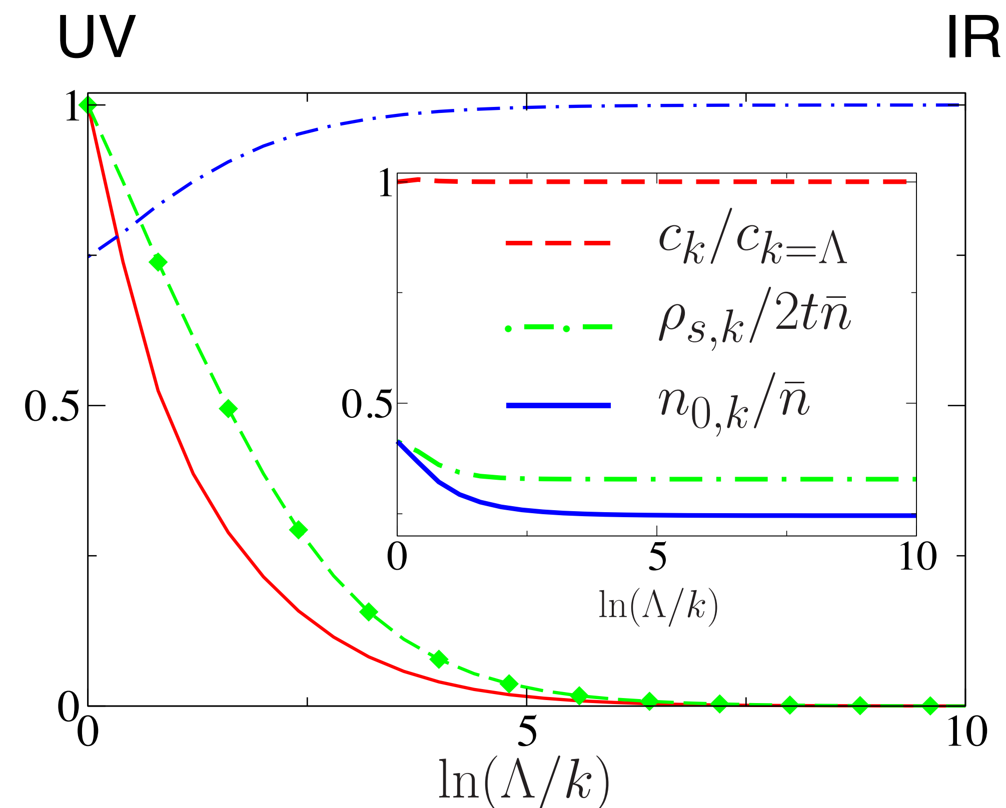
$$p_{np} \sim \sqrt{\bar{n}(U/t)^3}$$

crossover from linear to Ginzburg regime

- strongly correlated superfluid: Absence of Bogoliubov regime



weakly correlated superfluid



strongly correlated superfluid

running quantities: $p_h = \sqrt{\bar{n}(U/t)}$

inset: physical observables: speed of sound, superfluid fraction, condensate order parameter

main figure: running couplings from the truncation

Critical Behavior in the Bose-Hubbard Model

- We give a symmetry argument for a "bicritical" point with different dynamical exponent at the tip of the lobe
  - The full effective action (including fluctuations) at low energies has a derivative expansion

$$\Gamma[\psi] = \int \psi^*[Z\partial_\tau + Y\partial_\tau^2 + m^2 + ...]\psi + \lambda(\psi^*\psi)^2 + ...$$

  - At the phase transition, we have $m^2 = 0$. At the tip of the lobe, we have additionally(vertical tangent)

$$\frac{\partial m^2}{\partial \mu} = 0$$

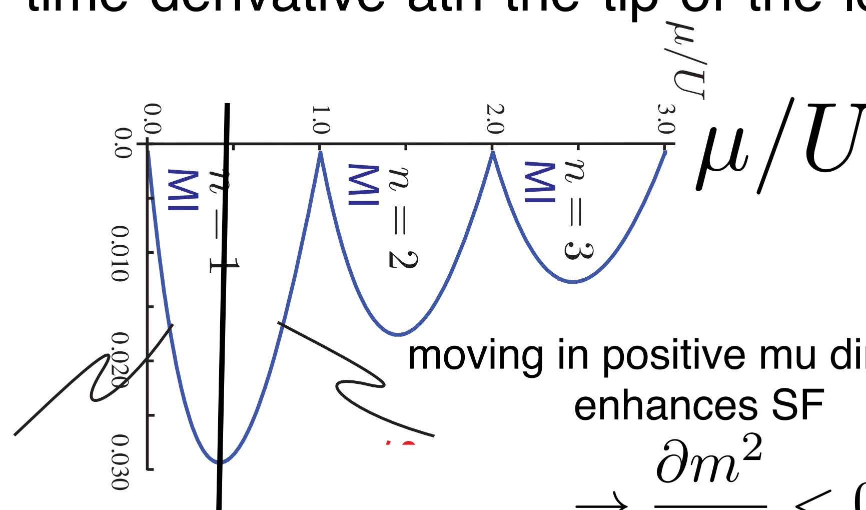
  - Using the invariance under temporally the local symmetry$\psi \to \psi e^{\mathrm{i}\theta(\tau)}, \mu \to \mu + \mathrm{i}\partial_\tau\theta(\tau)$, we find the Ward identity ($q = (\omega, \mathbf{q})$)

$$-\frac{\partial m^2}{\partial \mu} = -\frac{\partial}{\partial \mu}\frac{\delta^2\Gamma}{\delta\psi^*(q)\delta\psi(q)}\Big|_{\psi=0;q=0} = \frac{\partial}{\partial(\mathrm{i}\omega)}\frac{\delta^2\Gamma}{\delta\psi^*(q)\delta\psi(q)}\Big|_{\psi=0;q=0} = Z$$

  - Thus, there cannot be a linear time derivative ath the tip of the lobe, $Z = 0$. The leading frequency dependence is quadratic

moving in positive mu direction
suppresses SF

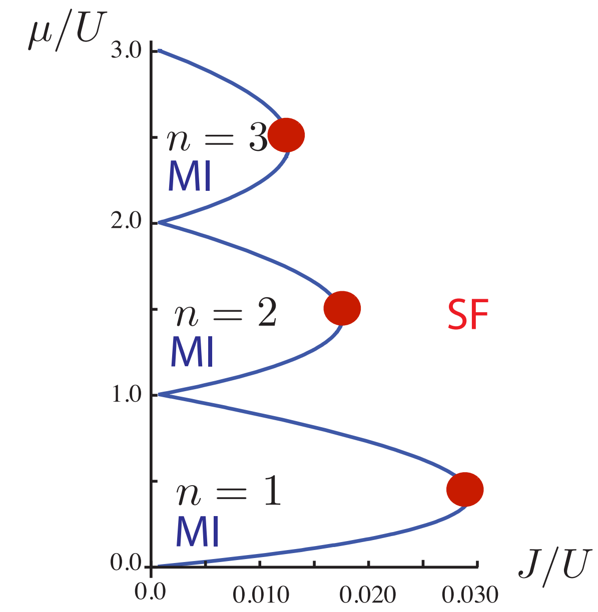
$$\rightarrow \frac{\partial m^2}{\partial \mu} > 0$$



moving in positive mu direction
enhances SF

$$\rightarrow \frac{\partial m^2}{\partial \mu} < 0$$

Bicritical Point from FRG



- As a consequence, we have the following critical behaviors
  - At generic points on the phase boundary the dispersion is nonrelativistic

dynamical exponent

$$\omega \sim \mathbf{q}^2 \qquad \Rightarrow z = 2$$

  - At the tip of the lobe, by symmetry the dispersion is relativistic

$$\omega \sim |\mathbf{q}| \qquad \Rightarrow z = 1$$

  - the effective dimension obtains from the power counting:

$$d_{\text{eff}} = d + z$$

  - the upper critical dimension, where mean field behavior is expected, is

$$d_{\text{crit},+} = 4$$

  - established in FRG analysis for d=2 from microscopic model:

    - generic points: mean field like critical behavior (log corrections)

$$d_{\text{eff}} = 4$$

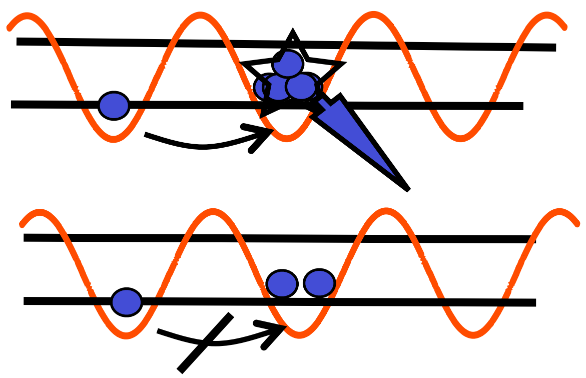
    - tip of the lobe: critical behavior of O(2) model in

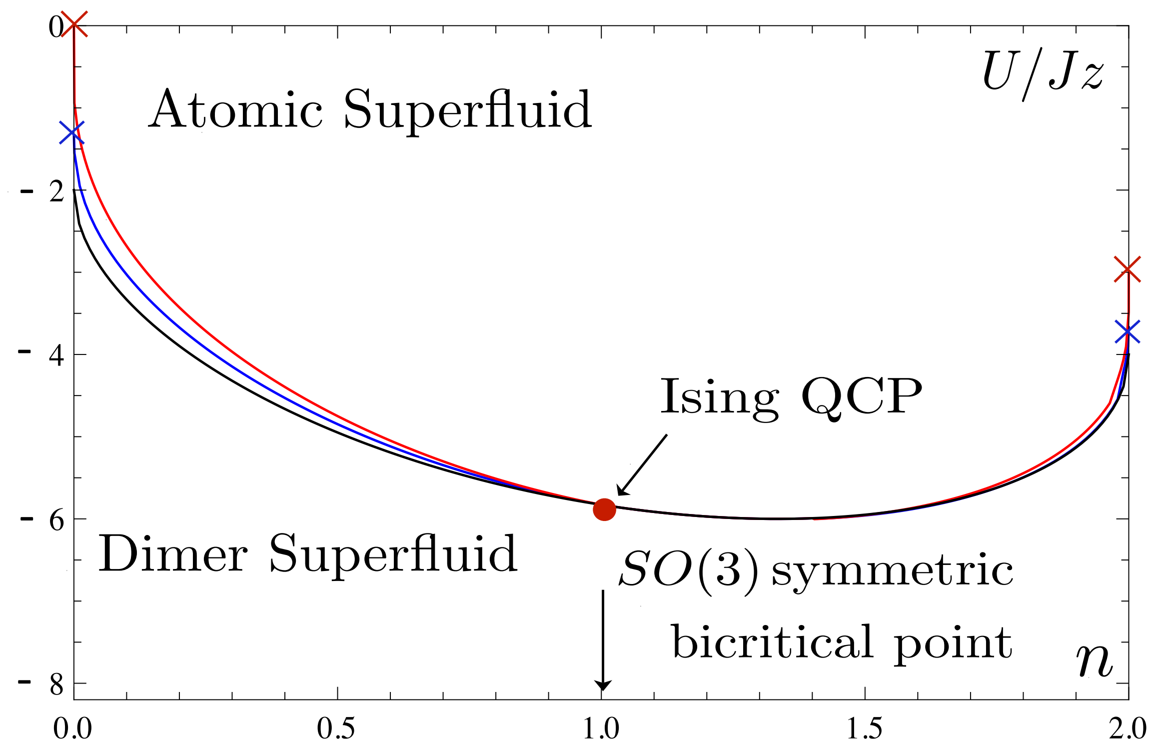
$$d_{\text{eff}} = 3$$

	e.g. anomalous dimension	crit. exponent
FRG tip BH model	$\eta = 0.049$	$\nu = 0.699$
high precision estimate	$\eta = 0.038$	$\nu = 0.671$

(A. Rancon & N. Dupuis, arxiv:1012.0166)

Attractive Lattice Bosons with Three-Body Constraint





Motivation

- Remember fermions: BCS-BEC crossover (not: quantum phase transition), since

$$\langle \psi_\uparrow \rangle = \langle \psi_\downarrow \rangle = 0 \qquad \text{Pauli principle}$$

$$\langle \psi_\uparrow \psi_\downarrow \rangle \neq 0 \qquad \text{pairing order}$$

- This is different for bosons: two symmetry breaking patterns may occur

$$\langle \hat{b} \rangle \neq 0, \quad \langle \hat{b}^2 \rangle \neq 0$$ - Conventional SF

$$\langle \hat{b} \rangle = 0, \quad \langle \hat{b}^2 \rangle \neq 0$$ - "Dimer SF"

- Thus, in a bosonic analog of the crossover problem, there should be a quantum phase transition, reminiscent of an Ising transition, since (cf Radzihovsky& '03; Stoof, Sachdev& '03):

$$\langle \hat{b} \rangle \sim \exp i\theta \qquad \langle \hat{b}^2 \rangle \sim \exp 2i\theta$$

➡ Spontaneous breaking of Z_2 symmetry $\theta \longrightarrow \theta + \pi$ of the DSF order parameter

0 Ising QPT? $(ak_F)^{-1}$

- The phase transition should be seen upon increasing the boson attraction (molecule formation)
- Problem: attractive bosons are unstable towards collapse (they seek the solid ground state)

Stabilizing Attractive Bosons

- Problem: attractive bosons are unstable towards collapse (they seek the solid ground state)
- on the lattice, one could imagine a situation with two-body attraction but three-body repulsion:

$$H = -J \sum_{\langle i,j \rangle} b_i^\dagger b_j - \mu \sum_i \hat{n}_i + \frac{1}{2} U \sum_i \hat{n}_i(\hat{n}_i - 1) + \frac{1}{6} V \sum_i \hat{n}_i(\hat{n}_i - 1)(\hat{n}_i - 2)$$

- for

$$U < 0 \qquad V > 0 \qquad\qquad V/|U| \gg 1$$

- attractive two-body forces
- three-body interaction acts as a constraint against three-fold and higher local occupation: stabilize against collapse

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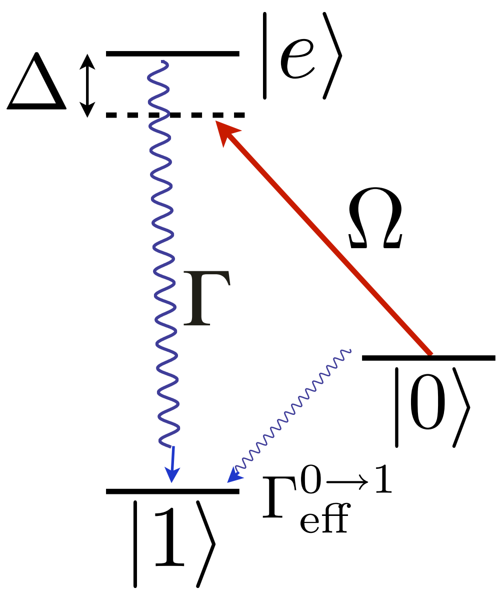
- There is a dissipative mechanism based on strong three-body loss which realizes such a three-body hardcore constraint

An analogy: optical pumping

A. Daley, J. Taylor, SD, P. Zoller '09



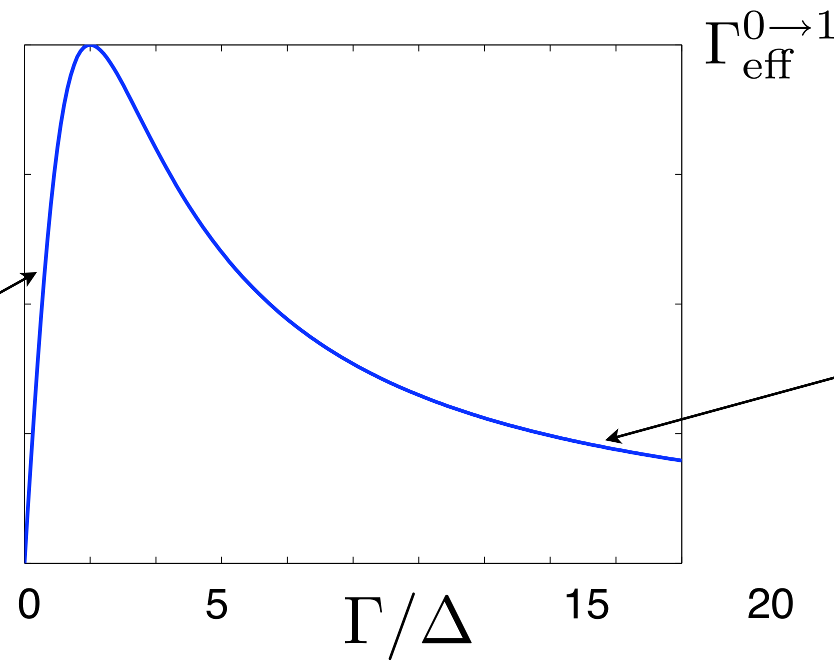
master equation in Lindblad form

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = -i[H,\rho] + \mathcal{L}[\rho]$$

with $$H = \frac{\Omega}{2}(|0\rangle\langle e| + |e\rangle\langle 0|) - \Delta|e\rangle\langle e|$$

$$\mathcal{L}[\rho] = \Gamma\left(J\rho J^\dagger - \frac{1}{2}(J^\dagger J\rho + \rho J^\dagger J)\right) \qquad J = |1\rangle\langle e|$$

pumping rate $$\Gamma_{\text{eff}}^{0\to1} = \frac{\Omega^2}{4\Delta^2+\Gamma^2}\Gamma \qquad (\text{for } \Omega \ll \Gamma, \Delta)$$



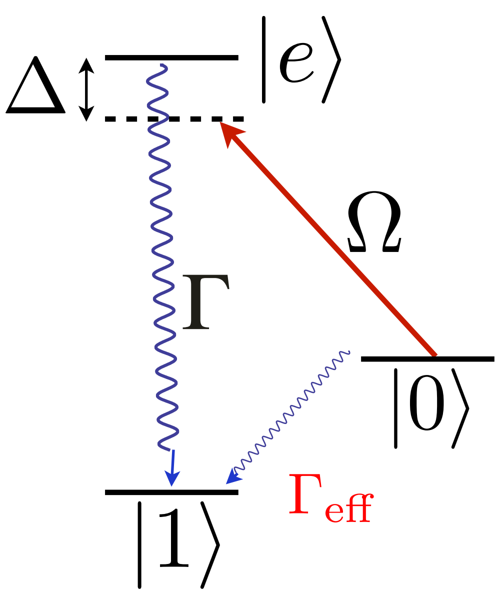
$$\Gamma_{\text{eff}}^{0\to1} \approx \frac{\Omega^2}{4\Delta^2}\Gamma$$

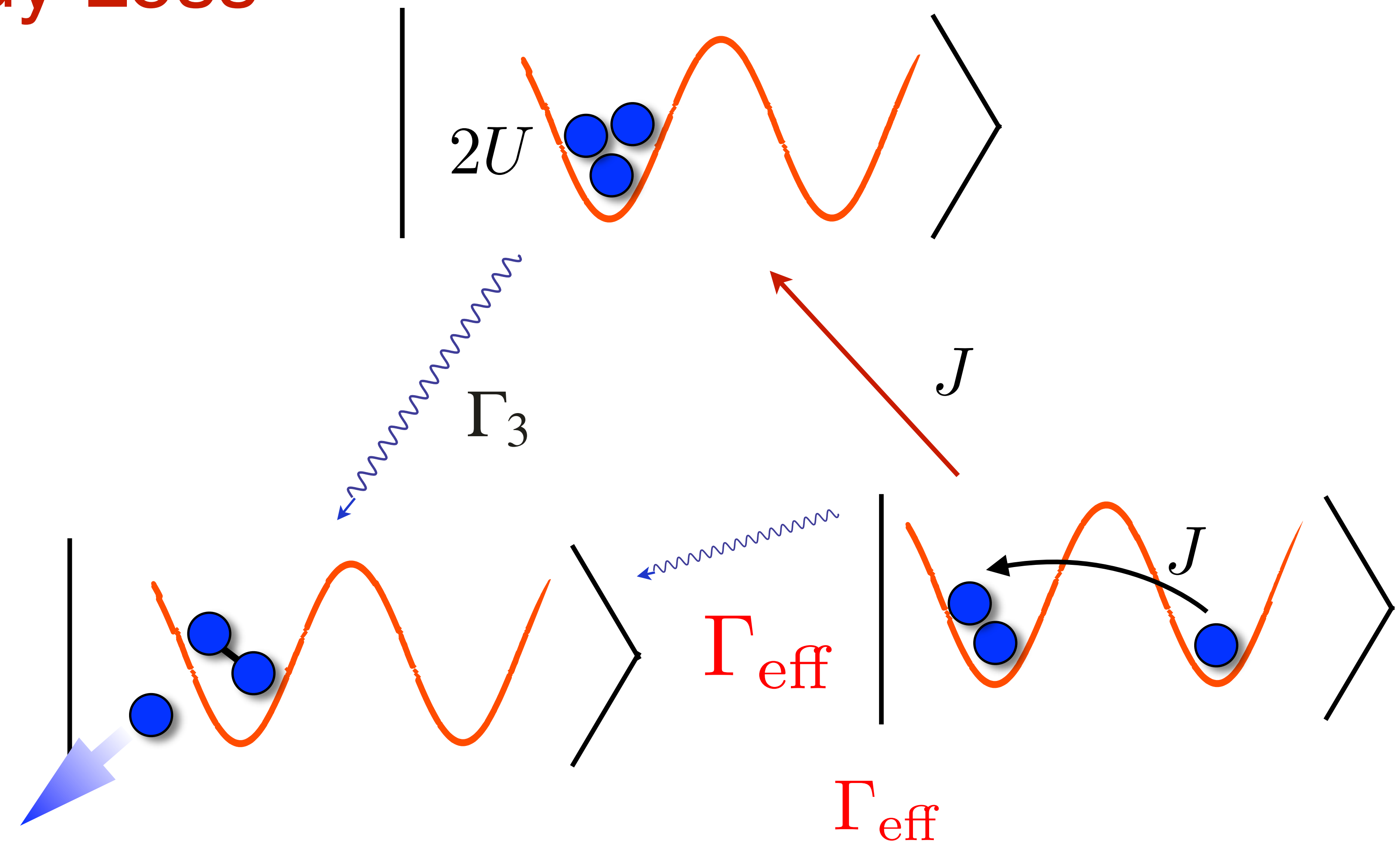
$$\Gamma_{\text{eff}}^{0\to1} \approx \frac{\Omega^2}{\Gamma}$$

Zeno regime: system frozen in $|0\rangle$

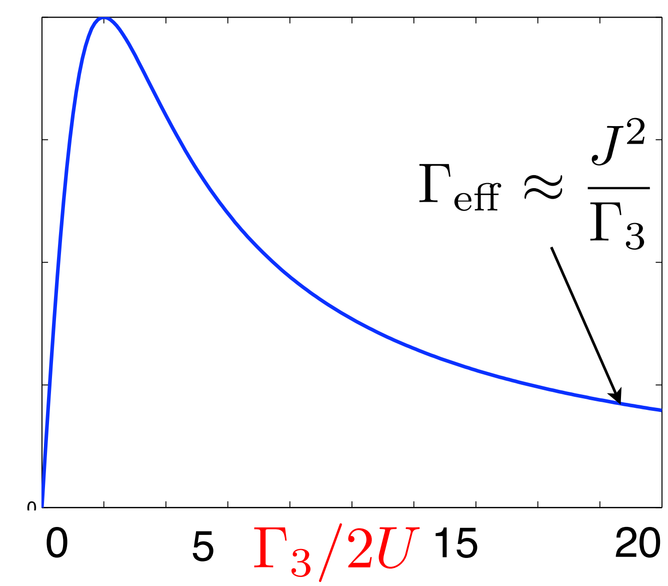
large Γ:
- system "freezes" in I0>
- leading virtual process is effective small loss rate 0 -> 1

Analogy to Three-Body Loss





detuning	Δ	$\longleftrightarrow$	$2U$	onsite interaction energy
Rabi frequency	Ω	$\longleftrightarrow$	J	tunnel coupling
decay rate	Γ	$\longleftrightarrow$	Γ_3	three-body recombination rate



- system "freezes" in subspace with fewer 3 particles per site: 3-body constraint
- stabilizes against particle loss: effective loss rate $\Gamma_{\text{eff}} \approx J^2/\Gamma_3$

➡ For $\Gamma_3 \gg U, J$, realization of a Bose-Hubbard-Hamiltonian with three-body hard-core constraint on time scales $t < 1/\Gamma_{\text{eff}}$

Analyzing Constrained Lattice Bosons

(SD, M. Baranov. A. Daley, P. Zoller '09,'10)

- There is a simple mean field theory (Gutzwiller factorization of the ground state wave function)
- but it misses out physics at various length scales:

microscopic	thermodynamic	long distance
$\varepsilon_M = -\frac{1}{Ma^2}$	$n = \frac{k_F^3}{3\pi^2}, T$	$k_{ld} \gg n^{1/3}, T^{1/2}, \varepsilon_M^{1/2}$
bound state formation	shifts in the phase border	nature of the phase transition

see appendix for details!

- How to go beyond?
  - MFT is a classical field theory
  - Find a means to requantize this MFT: classical field theory -> quantum field theory
    - exact mapping of the constrained Hamiltonian to a coupled boson theory with polynomial interactions
    - the bosonic operators find natural interpretation in terms of "atoms" and "dimers"

$PHP \rightarrow$ "projected" Bose-Hubbard Hamiltonian (with constraint)

$X_i = 1 - \hat{n}_{1,i} - \hat{n}_{2,i}$ "atoms" "dimers"

$$H = (U - 2\mu)\sum_i \hat{n}_{2,i} - \mu \sum_i \hat{n}_{1,i} - J \sum_{\langle i,j\rangle} \left[t^\dagger_{1,i} X_i X_j t_{1,j} + \sqrt{2}(t^\dagger_{2,i} t_{1,i} X_j t_{1,j} + t^\dagger_{1,i} X_i t^\dagger_{1,j} t_{2,j}) + 2 t^\dagger_{2,i} t_{2,j} t^\dagger_{1,j} t_{1,i} \right]$$

➡ We have identified several quantitative and qualitative effects:

✓Tied to interactions

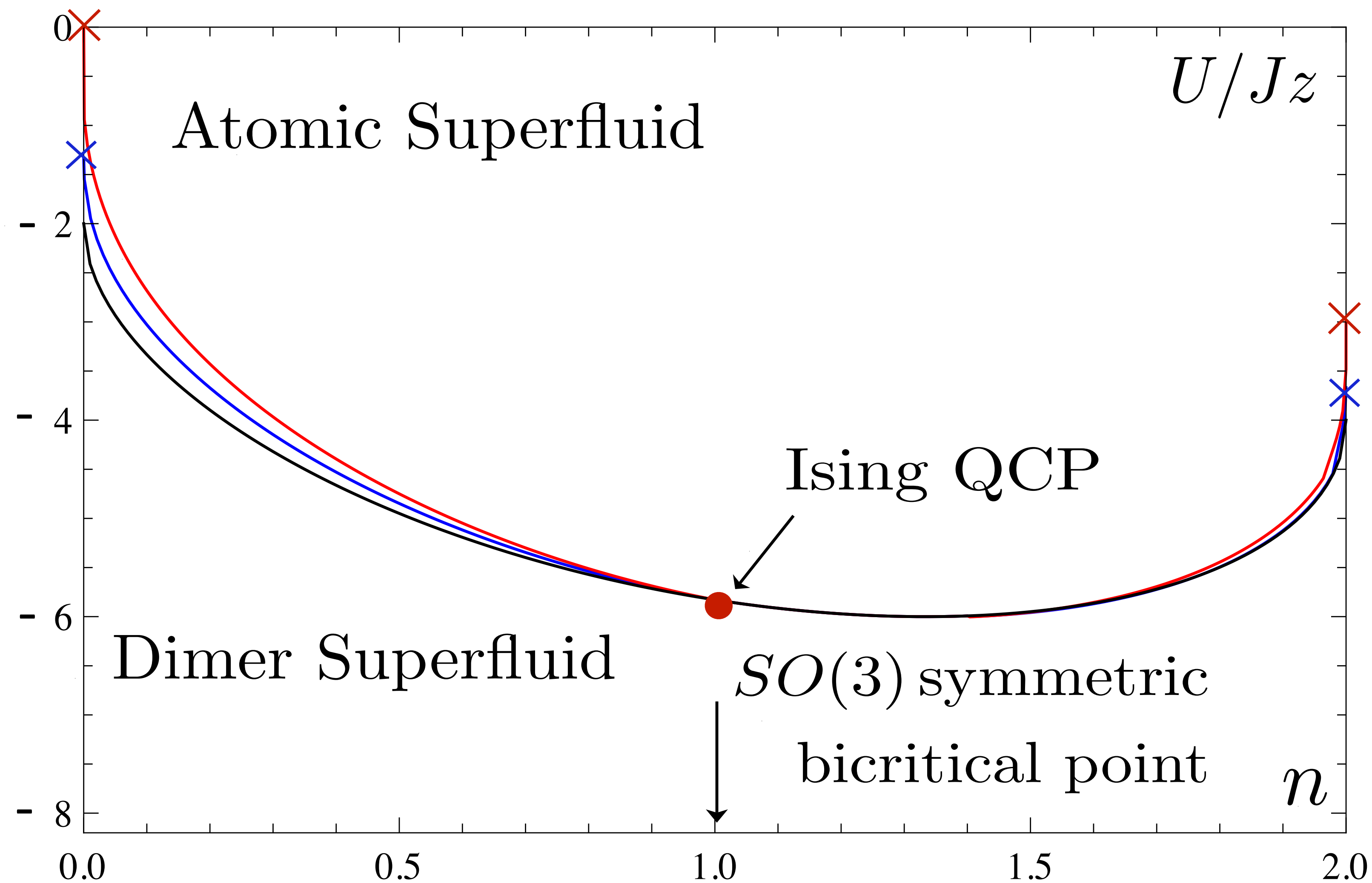
✓Tied to the constraint

Beyond Mean Field Phase Diagram

(SD, M. Baranov. A. Daley, P. Zoller ‘09,’10)

• Qualitative effects of the constraint and interactions:

Phase diagram for attractive boson with three-body constraint



➡ Enhancement of symmetry from $SO(2) \sim U(1) \rightarrow SO(3)$

➡ Ising quantum critical point near half filling

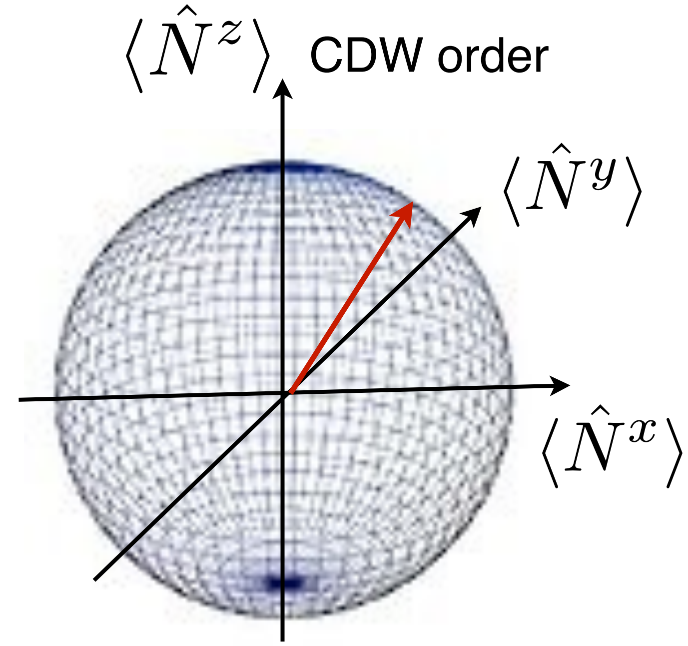
microscopic $\varepsilon_M = -\frac{1}{Ma^2}$ → thermodynamic $n = \frac{k_F^3}{3\pi^2}, T$ → long distance $k_{ld} \gg n^{1/3}, T^{1/2}, \varepsilon_M^{1/2}$

Symmetry Enhancement in Strong Coupling

- Perturbative limit U >> J: expect dimer hardcore model
- Interpret EFT as a spin 1/2 model in external field:

$$H_{\text{eff}} = -2t \sum_{\langle i,j \rangle} \left(s_i^x s_j^x + s_i^y s_j^y + \lambda s_i^z s_j^z \right) \qquad t = \frac{2J^2}{|U|}$$

- Leading (second) order perturbation theory: $\boxed{\lambda = \frac{v}{2t} = 1}$



xy plane: superfluid order

➡ Isotropic Heisenberg model (half filling n=1):

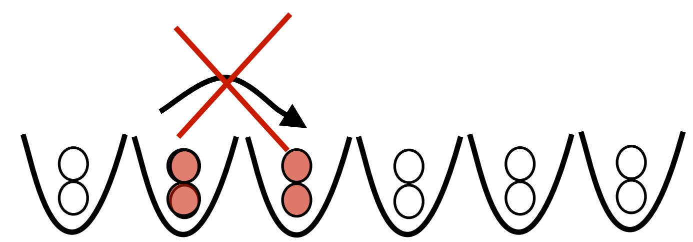
- Emergent symmetry: SO(3) rotations vs. SO(2) sim U(1)
- Bicritical point with Neel vector order parameter

$$\hat{N}^\alpha = \sum_j (-)^j s_i^\alpha$$

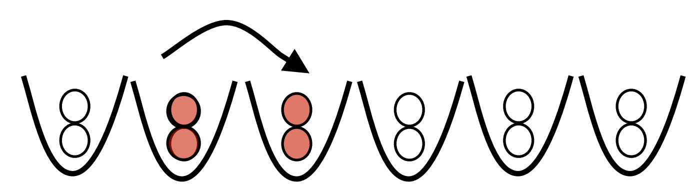
- charge density wave and superfluid exactly degenerate
  - CDW: Translation symmetry breaking
  - DSF: Phase symmetry breaking
- physically distinct orders can be freely rotated into each other:

"continuous supersolid"

with constraint $\lambda = 1$



without constraint $\lambda = 4$



➡ The symmetry enhancement is unique to the 3-body hardcore constraint

Signatures of "continuous supersolid"

- Next (fourth) order perturbation theory: Superfluid preferred

$$\lambda = 1 - 8(z-1)(J/|U|)^2 < 1$$

- Proximity to bicritical point governs physics in strong coupling

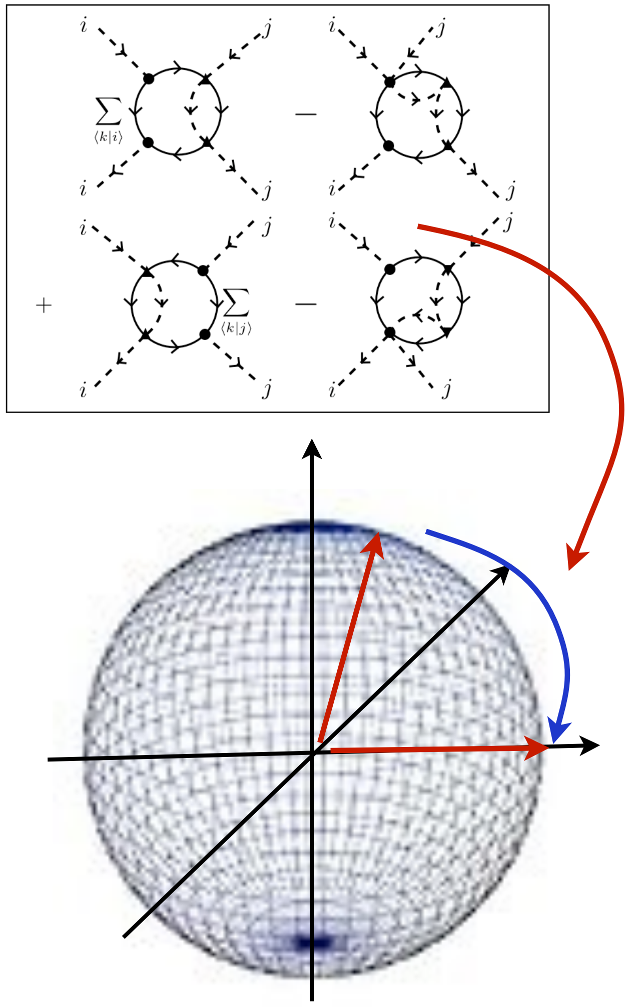
(1) Second collective (pseudo) Goldstone mode

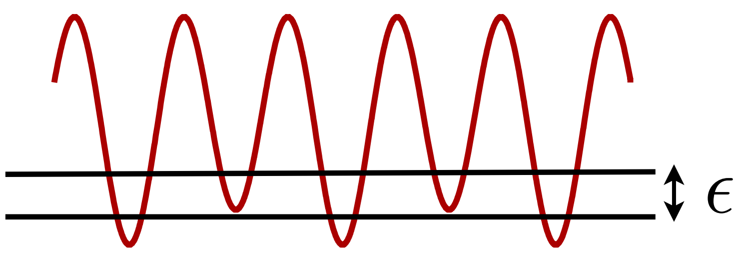
$$\omega(\mathbf{q}) = tz\big((\lambda\varepsilon_{\mathbf{q}} + 1)(1-\varepsilon_{\mathbf{q}})\big)^{1/2}$$

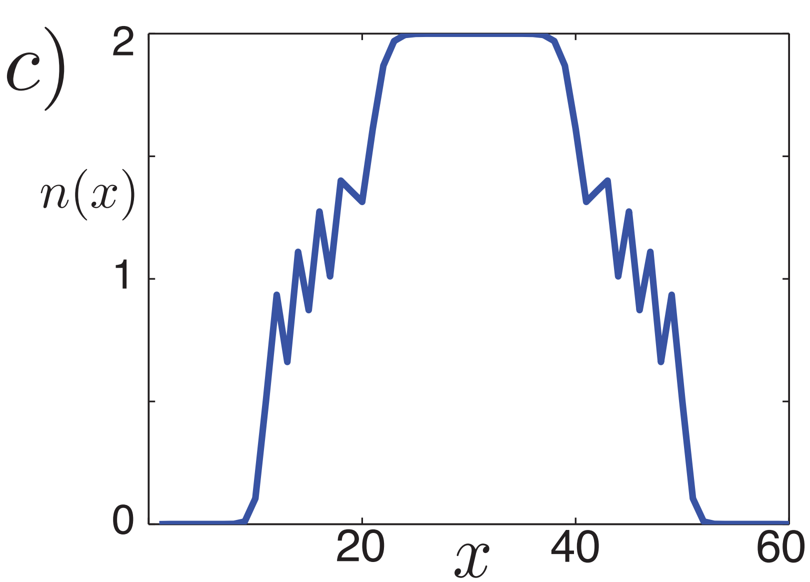
(2) Use weak superlattice to rotate Neel order parameter

$$\epsilon/tz = \Delta/tz = 1 - \lambda \approx 8(z-1)(J/U)^2$$

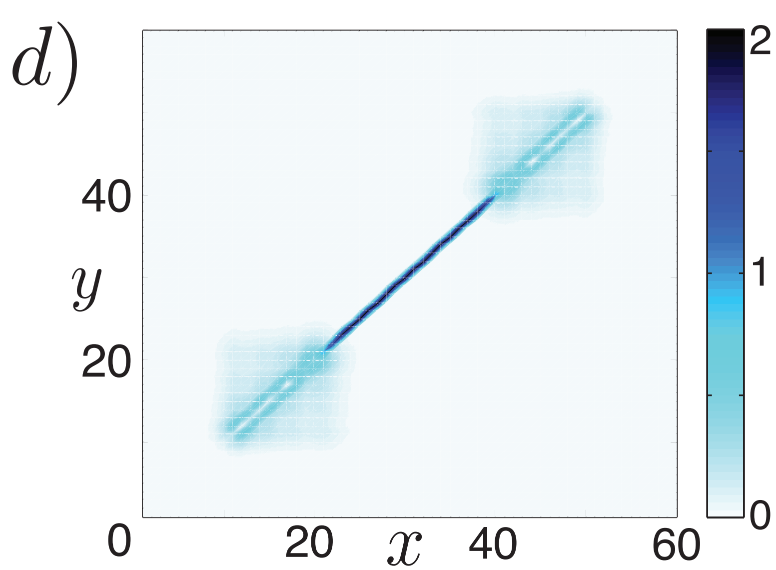
(3) Simulation of 1D experiment in a trap (t-DMRG)



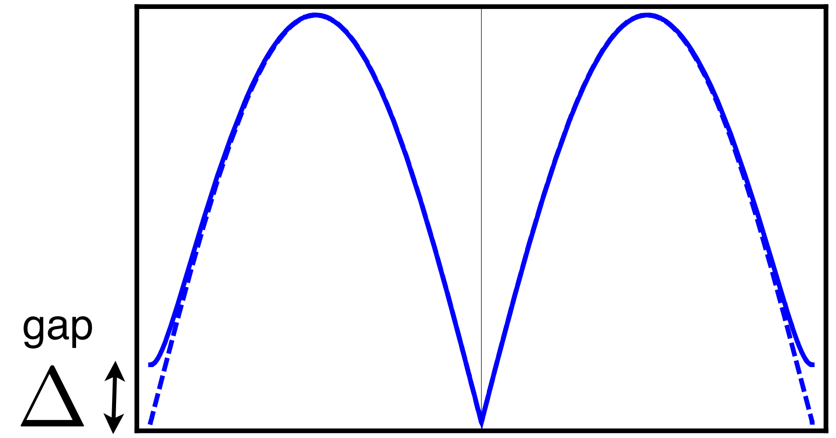




density profile: Onset of CDW



DSF order in textured regions



Second (pseudo) Goldstone mode

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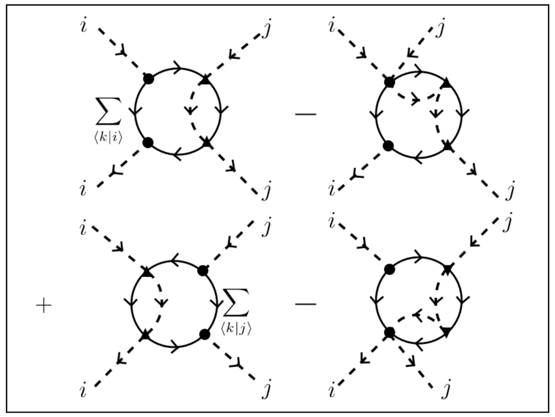
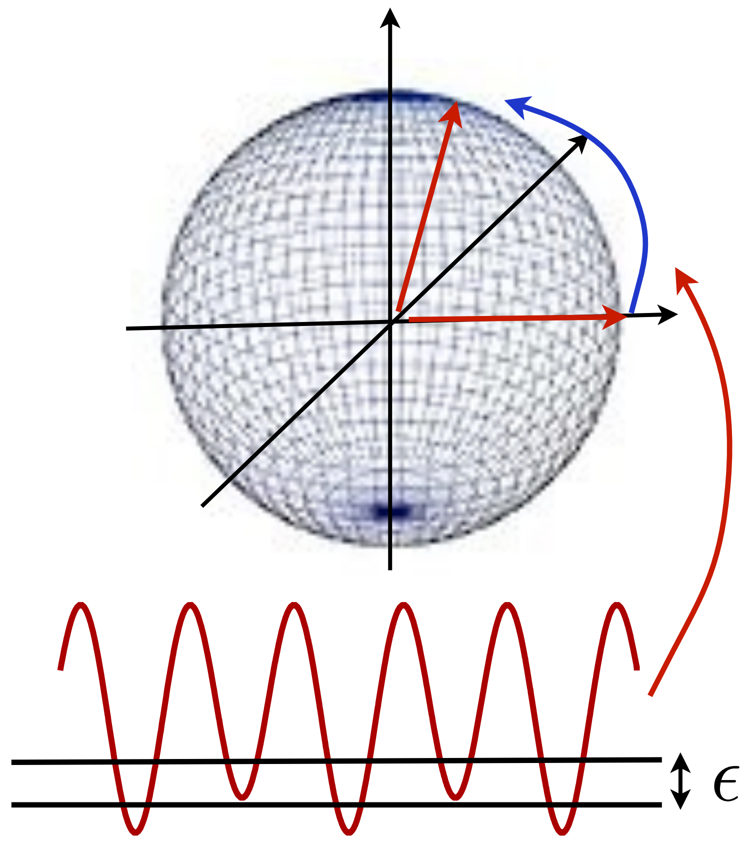
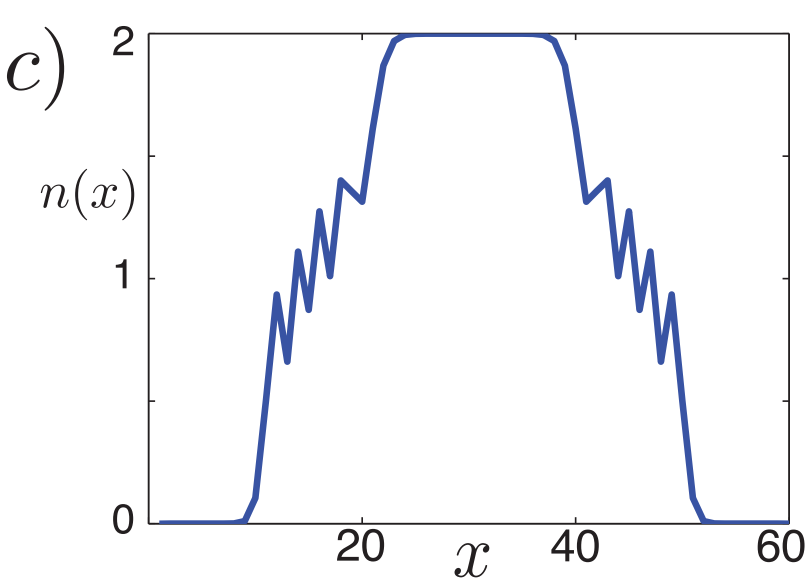
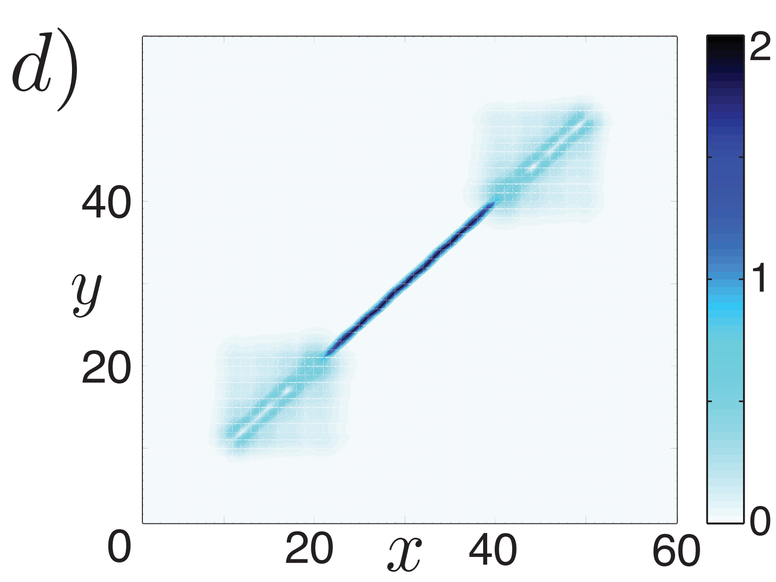


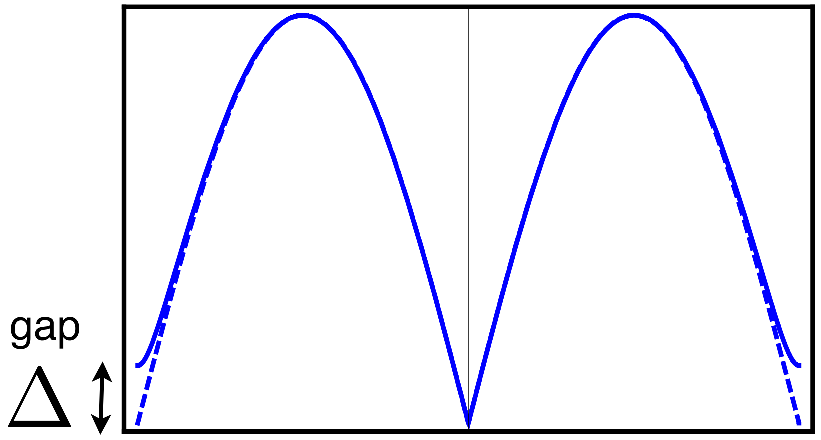




density profile: Onset of CDW



DSF order in textured regions



Second (pseudo) Goldstone mode

Infrared Limit: Nature of the Phase Transition

- Two near massless modes: Critical atomic field, dimer Goldstone mode
- Coleman-Weinberg phenomenon for coupled real fields: Radiatively induced first order PT
- Perform the continuum limit and integrate out massive modes:

pure Goldstone action

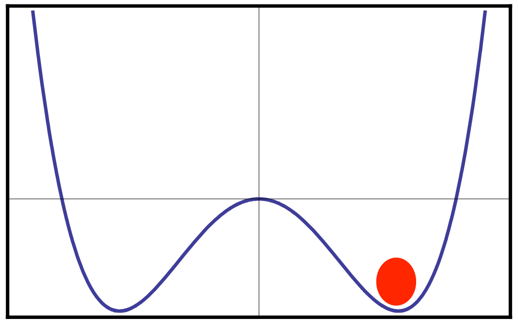
$$S[\vartheta,\phi] = S_{\mathrm{I}}[\phi] + S_{\mathrm{G}}[\vartheta] + S_{\mathrm{int}}[\vartheta,\phi]$$

pure Ising action

coupling term

$$S_{\mathrm{I}}[\phi] = \int \partial_\mu\phi\partial^\mu\phi + m^2\phi^2 + \lambda\phi^4$$

Ising field: Real part of atomic field



Ising potential landscape:
Z_2 symmetry breaking

$$S_{\mathrm{int}}[\vartheta,\phi] = \mathrm{i}\kappa \int \partial_\tau\vartheta\,\phi^2$$

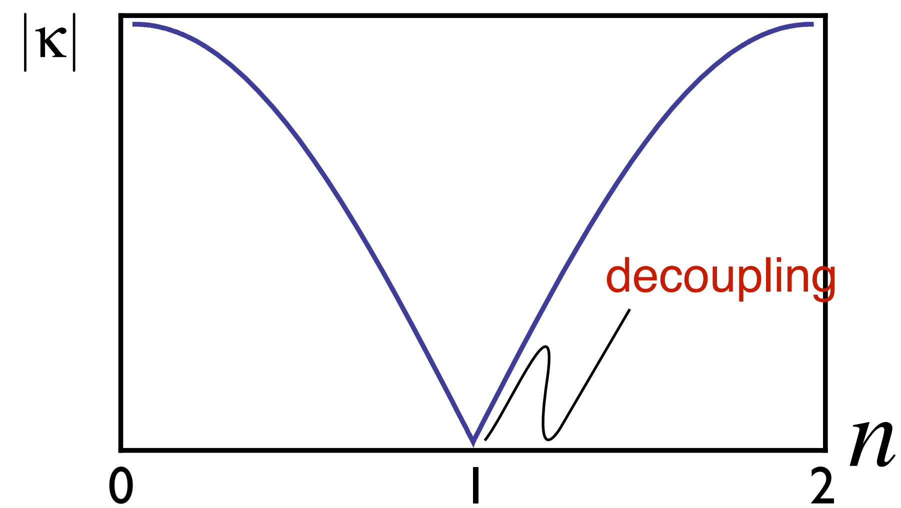
Frey, Balents; Radzihovsky&

➡ Interactions persist to arbitrary long wavelength (cf. decoupling spin waves)

➡ $\kappa \neq 0$ Phase transition is driven first order by coupling of Ising and Goldstone mode

Ising Quantum Critical Point around n=1

- Plot the Ising-Goldstone coupling:



$$S_{\mathrm{int}}[\vartheta, \phi] = \mathrm{i}\kappa \int \partial_\tau \vartheta \, \phi^2$$

$$\Gamma \ni \int_{\vec{x},\tau} b_{2,i}^{\dagger}(-g_2 \mu) b_{2,i}$$

- Symmetry argument:
  - analysis of limiting cases n -> 0, n -> 2 and continuity: dimer compressibility must have zero crossing
  - Ward identies for time-local gauge invariance and atom-dimer phase locking

  ➡ κ must have zero crossing: true quantum critical Ising transition

- Estimate correlation length: $\xi / a \sim \kappa^{-6} \sim |1-n|^{-6}$

  ➡weakly first order, broad near critical domain

➡ Second order quantum critical behavior is a lattice + constraint effect

Appendix: Quantum Field Theory for Locally Constrained Lattice Models

Implementation of the Hard-Core Constraint

- Introduce operators to parameterize on-site Hilbert space

$$t^{\dagger}_{\alpha,i}|\mathrm{vac}\rangle = |\alpha\rangle, \quad \alpha = 0, 1, 2$$

- They are not independent:

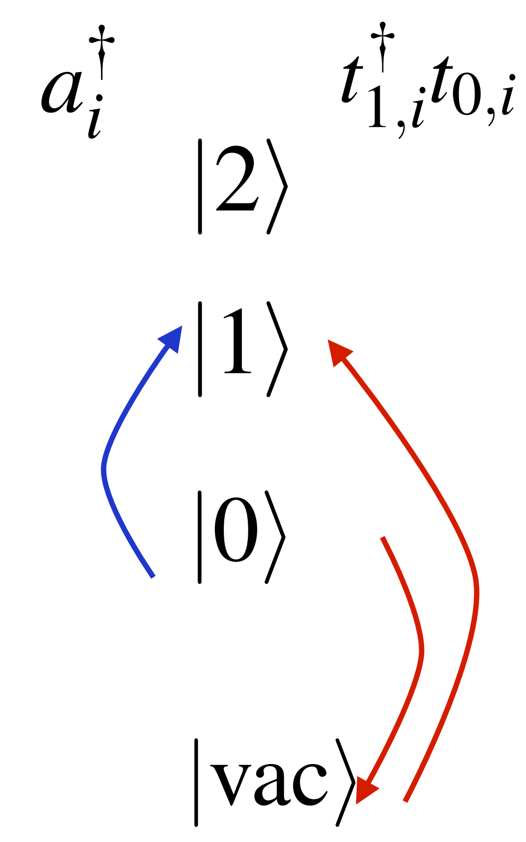
$$\sum_{\alpha} t^{\dagger}_{\alpha,i} t_{\alpha,i} = \mathbf{1}$$

- Representation of Hubbard operators:

$$a^{\dagger}_{i} = \sqrt{2}\, t^{\dagger}_{2,i} t_{1,i} + t^{\dagger}_{1,i} t_{0,i}$$

$$\hat{n}_i = 2 t^{\dagger}_{2,i} t_{2,i} + t^{\dagger}_{1,i} t_{1,i}$$

Action of operators



Implementation of the Hard-Core Constraint

- Hamiltonian:

$$H_{\text{pot}} = -\mu \sum_i 2t^\dagger_{2,i}t_{2,i} + t^\dagger_{1,i}t_{1,i} + U\sum_i t^\dagger_{2,i}t_{2,i}$$

$$H_{\text{kin}} = -J\sum_{\langle i,j\rangle}\left[t^\dagger_{1,i}t_{0,i}t^\dagger_{0,j}t_{1,j} + \sqrt{2}(t^\dagger_{2,i}t_{1,i}t^\dagger_{0,j}t_{1,j} + t^\dagger_{1,i}t_{0,i}t^\dagger_{1,j}t_{2,j}) + 2t^\dagger_{2,i}t^\dagger_{1,j}t_{1,i}t_{2,j}\right]$$

- Properties:
  - Mean field: Gutzwiller energy (classical theory)
  - interaction: quadratic
  - hopping: higher order
    - Role of interaction and hopping reversed
    - Strong coupling approach facilitated
  - One phase is redundant: absorb via *local* gauge transformation

$$t_{1,i} = \exp \mathrm{i}\varphi_{0,i}\, |t_{0,i}| \qquad\qquad t_{1,i} \to \exp -\mathrm{i}\varphi_{0,i}\, t_{1,i}, \quad t_{2,i} \to \exp -\mathrm{i}\varphi_{0,i}\, t_{2,i}$$

➡ e.g. t_0 can be chosen real

Implementation of the Hard-Core Constraint

- Resolve the relation between t-operators (zero density) (SD, M. Baranov, A. Daley, P. Zoller '09, '10)

$$\boxed{t^\dagger_{1,i} t_{0,i} = t^\dagger_{1,i}\sqrt{1 - t^\dagger_{1,i} t_{1,i} - t^\dagger_{2,i} t_{2,i}} \rightarrow t^\dagger_{1,i}(1 - t^\dagger_{1,i} t_{1,i} - t^\dagger_{2,i} t_{2,i})}$$

  - justification: for projective operators one has from Taylor representation

$$X^2 = X \rightarrow f(X) = f(0)(1 - X) + X f(1) \qquad X = 1 - t^\dagger_{1,i} t_{1,i} - t^\dagger_{2,i} t_{2,i}$$

- Now we can interpret the remaining operators as standard bosons:

  - on-site bosonic space $\qquad \mathcal{H}_i = \{|n\rangle^1_i |m\rangle^2_i\}, \quad n, m = 0, 1, 2, \ldots$

  - decompose into physical/unphysical space: $\qquad \mathcal{H}_i = \mathcal{P}_i \oplus U_i$

$$\mathcal{P}_i = \{|0\rangle^1_i |0\rangle^2_i, |1\rangle^1_i |0\rangle^2_i, |0\rangle^1_i |1\rangle^2_i\}$$

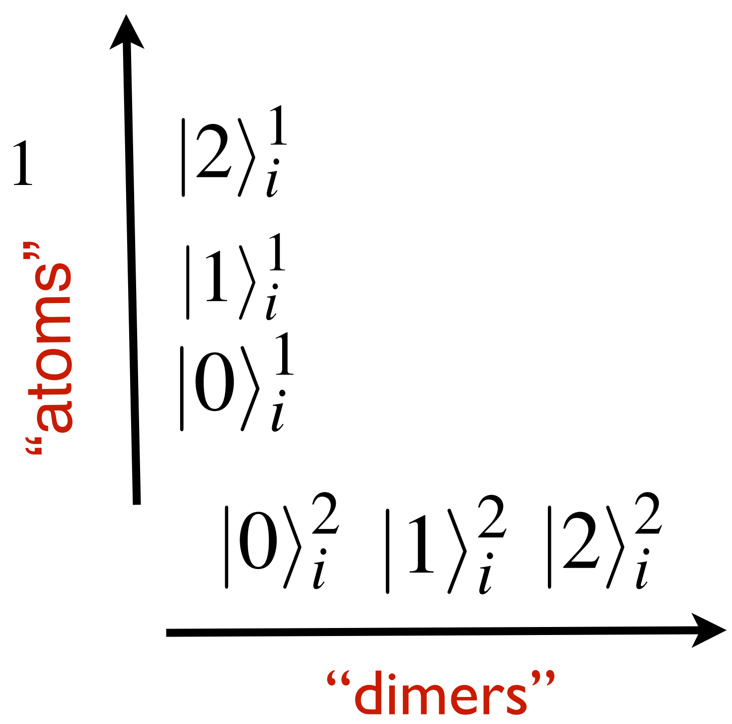
  - correct bosonic enhancement factors on physical subspace $\qquad \sqrt{n} = 0, 1$

  - the Hamiltonian is an involution on P and U:

$$H = H_{PP} + H_{UU}$$

  - remaining degrees of freedom: "atoms" and "dimers"

➡ similarity to Hubbard-Stratonovich transformation



Implementation of the Hard-Core Constraint

- The partition sum does not mix U and P too:

$$Z = Tr \exp -\beta H = Tr_{PP} \exp -\beta H_{PP} + Tr_{UU} \exp -\beta H_{UU}$$

- Need to discriminate contributions from U and P: Work with Effective Action

  - Legendre transform of the Free energy $\quad W[J] = \log Z[J]$

$$\Gamma[\chi] = -W[J] + \int J^T \chi, \quad \chi \equiv \frac{\delta W[J]}{\delta J}$$

Quantum Equation of Motion for J=0

  - Has functional integral representation:

$$\exp -\Gamma[\chi] = \int \mathcal{D}\delta\chi \exp -S[\chi + \delta\chi] + \int J^T \delta\chi, \quad J = \frac{\delta\Gamma[\chi]}{\chi}$$

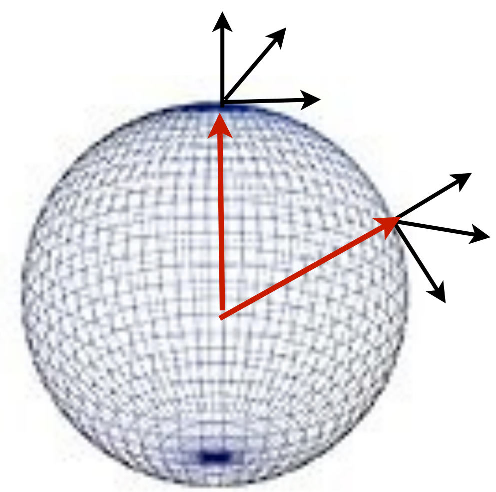
$$S[\chi = (t_1, t_2)] = \int d\tau \Big(\sum_i t^\dagger_{1,i} \partial_\tau t_{1,i} + t^\dagger_{2,i} \partial_\tau t_{2,i} + H[t_1, t_2] \Big)$$

- Usually: Effective Action shares all symmetries of S
- Here: symmetry principles are supplemented with a constraint principle

Condensation and Thermodynamics

- Physical vacuum is continuously connected to the finite density case:

Introduce new, expectationless operators by (complex) Euler rotation



$$\vec{t} = (t_0, t_1, t_2)^T$$

$$\vec{b} = R_\theta \, R_\varphi \, \vec{t}$$

- Hamiltonian in new coordinates takes form:

Quadratic part: Spin waves (Goldstone for n > 0)

$$H = E_{\mathrm{GW}} + H_{\mathrm{SW}} + H_{\mathrm{int}}$$

Mean field: Gutzwiller Energy

higher order: interactions

microscopic

interactions

thermodynamic

interactions
condensation

long distance

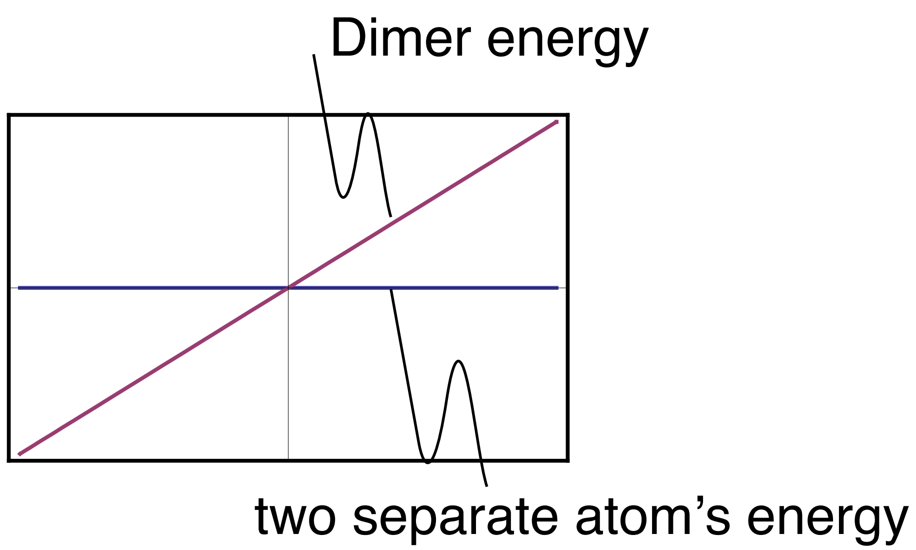
interactions
condensation
spin waves

The requantized Gutzwiller model

- Hamiltonian to cubic order is of Feshbach type:
  - quadratic part:

$$H_{\text{pot}} = \sum_i (U - 2\mu) n_{2,i} - \mu n_{1,i}$$

detuning from atom level



  - leading interaction:

$$H_{\text{kin}} = -J \sum_{\langle i,j \rangle} \left[t^\dagger_{1,i} t_{1,j} + \sqrt{2} (t^\dagger_{2,i} t_{1,i} t_{1,j} + t^\dagger_{1,i} t^\dagger_{1,j} t_{2,j}) \right]$$

(bilocal) dimer splitting into atoms

- Compare to standard Feshbach models:

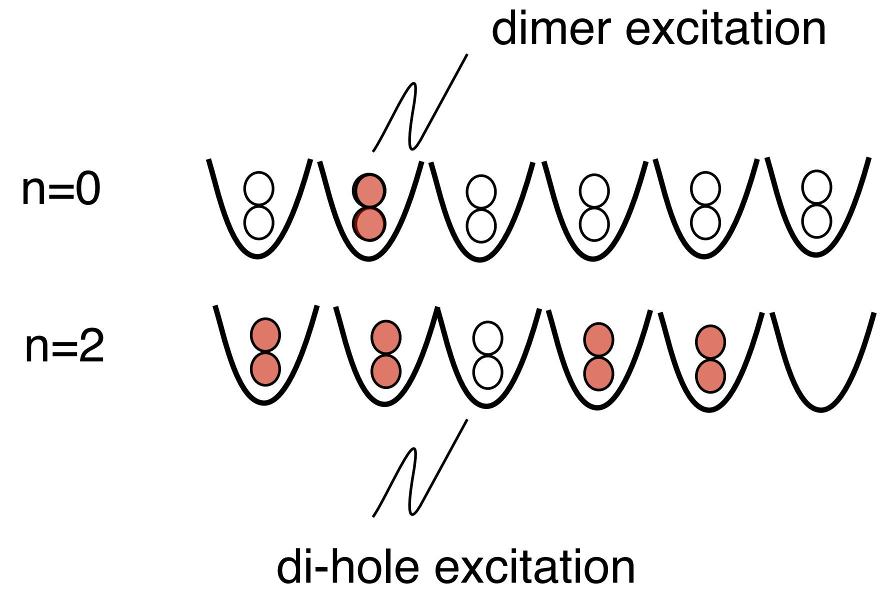
detuning $\sim 1/U$

here: detuning $\sim U$

➡ we can expect resonant (strong coupling) phenomenology at weak coupling

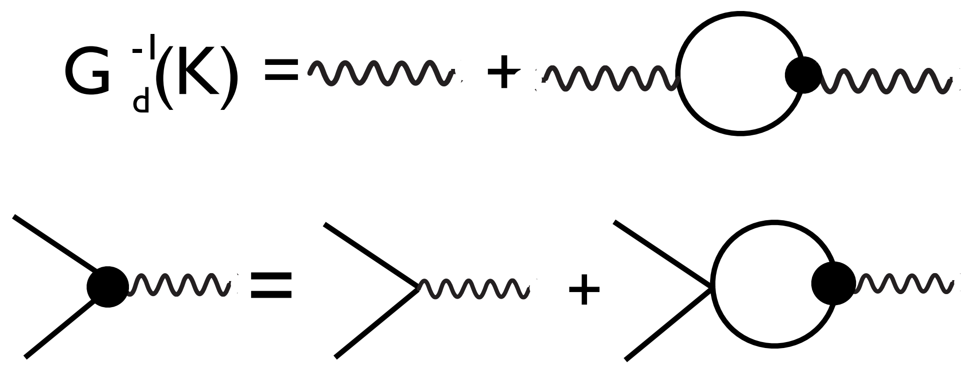
$$H_{\text{kin}} = -J \sum_{\langle i,j \rangle} \left[t^\dagger_{1,i} (1 - n_{1,i} - n_{2,i})(1 - n_{1,j} - n_{2,j}) t_{1,j} + \sqrt{2} (t^\dagger_{2,i} t_{1,i} (1 - n_{1,j} - n_{2,j}) t_{1,j} + t^\dagger_{1,i} (1 - n_{1,i} - n_{2,i}) t^\dagger_{1,j} t_{2,j}) + 2 t^\dagger_{2,i} t_{2,j} t^\dagger_{1,j} t_{1,i} \right]$$

Vacuum Problems



- The physics at n=0 and n=2 are closely connected:
  - "vacuum": no spontaneous symmetry breaking
  - low lying excitations:
    - n=0: atoms and dimers on the physical vacuum
    - n=2: holes and di-holes on the fully packed lattice

- Two-body problems can be solved exactly
  - Bound state formation: $G_d^{-1}(\omega = \mathbf{q} = 0) = 0$



$$\frac{1}{a_n|\tilde{U}| + b_n} = \int \frac{d^d q}{(2\pi)^d} \frac{1}{-\tilde{E}_b + c_n/d \sum_\lambda (1 - \cos \mathbf{q}\mathbf{e}_\lambda)}$$

$$n = 0: \quad a_0 = 1,\ b_0 = 0,\ c_0 = 2$$

➡ reproduces Schrödinger Equation: benchmark

➡ Square root expansion of constraint fails

$$n = 2: \quad a_2 = 4,\ b_2 = -6 + 3\tilde{E}_b\ c_2 = 4$$

➡ di-hole-bound state formation at finite U in 2D